0.4 FACTORING POLYNOMIALS

- Use special products and factorization techniques to factor polynomials.
- Find the domains of radical expressions.
- Use synthetic division to factor polynomials of degree three or more.
- Use the Rational Zero Theorem to find the real zeros of polynomials.

Factorization Techniques

The **Fundamental Theorem of Algebra** states that every nth-degree polynomial

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0$$

has precisely n **zeros.** (The zeros may be repeated or imaginary.) The problem of finding the zeros of a polynomial is equivalent to the problem of factoring the polynomial into linear factors.

Special Products and Factorization Techniques

Quadratic Formula

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example

$$x^2 + 3x - 1 = 0 \Rightarrow x = \frac{-3 \pm \sqrt{13}}{2}$$

Special Products

$$x^2 - a^2 = (x - a)(x + a)$$
$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$
$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$
$$x^4 - a^4 = (x - a)(x + a)(x^2 + a^2)$$

Examples

$$x^2 - 9 = (x - 3)(x + 3)$$
$$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$$
$$x^3 + 64 = (x + 4)(x^2 - 4x + 16)$$
$$x^4 - 16 = (x - 2)(x + 2)(x^2 + 4)$$

Binomial Theorem

$$(x + a)^2 = x^2 + 2ax + a^2$$
$$(x - a)^2 = x^2 - 2ax + a^2$$
$$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$$
$$(x - a)^3 = x^3 - 3ax^2 + 3a^2x - a^3$$
$$(x + a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$$
$$(x - a)^4 = x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4$$

Examples

$$(x + 3)^2 = x^2 + 6x + 9$$
$$(x^2 - 5)^2 = x^4 - 10x^2 + 25$$
$$(x + 2)^3 = x^3 + 6x^2 + 12x + 8$$
$$(x - 1)^3 = x^3 - 3x^2 + 3x - 1$$
$$(x + 2)^4 = x^4 + 8x^3 + 24x^2 + 32x + 16$$
$$(x - 4)^4 = x^4 - 16x^3 + 96x^2 - 256x + 256$$

$$(x + a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{2!}a^2x^{n-2} + \frac{n(n-1)(n-2)}{3!}a^3x^{n-3} + \cdots + na^{n-1}x + a^n{}^*$$
$$(x - a)^n = x^n - nax^{n-1} + \frac{n(n-1)}{2!}a^2x^{n-2} - \frac{n(n-1)(n-2)}{3!}a^3x^{n-3} + \cdots \pm na^{n-1}x \mp a^n$$

Factoring by Grouping

$$acx^3 + adx^2 + bcx + bd = ax^2(cx + d) + b(cx + d) = (ax^2 + b)(cx + d)$$

Example

$$3x^3 - 2x^2 - 6x + 4 = x^2(3x - 2) - 2(3x - 2) = (x^2 - 2)(3x - 2)$$

* The factorial symbol ! is defined as follows:
$0! = 1$, $1! = 1$, $2! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2 \cdot 1 = 6$, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, and so on.

EXAMPLE 1 Applying the Quadratic Formula

Use the Quadratic Formula to find all real zeros of each polynomial.

(a) $4x^2 + 6x + 1$ (b) $x^2 + 6x + 9$ (c) $2x^2 - 6x + 5$

SOLUTION

(a) Using $a = 4$, $b = 6$, and $c = 1$, you can write

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{36 - 16}}{8}$$
$$= \frac{-6 \pm \sqrt{20}}{8}$$
$$= \frac{-6 \pm 2\sqrt{5}}{8}$$
$$= \frac{2(-3 \pm \sqrt{5})}{2(4)}$$
$$= \frac{-3 \pm \sqrt{5}}{4}.$$

So, there are two real zeros:

$$x = \frac{-3 - \sqrt{5}}{4} \approx -1.309 \quad \text{and} \quad x = \frac{-3 + \sqrt{5}}{4} \approx -0.191.$$

(b) In this case, $a = 1$, $b = 6$, and $c = 9$, and the Quadratic Formula yields

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{36 - 36}}{2} = -\frac{6}{2} = -3.$$

So, there is one (repeated) real zero: $x = -3$.

(c) For this quadratic equation, $a = 2$, $b = -6$, and $c = 5$. So,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{36 - 40}}{4} = \frac{6 \pm \sqrt{-4}}{4}.$$

Because $\sqrt{-4}$ is imaginary, there are no real zeros.

STUDY TIP

Try solving Example 1(b) by factoring. Do you obtain the same answer?

TRY IT 1

Use the Quadratic Formula to find all real zeros of each polynomial.

(a) $2x^2 + 4x + 1$ (b) $x^2 - 8x + 16$ (c) $2x^2 - x + 5$

The zeros in Example 1(a) are irrational, and the zeros in Example 1(c) are imaginary. In both of these cases the quadratic is said to be **irreducible** because it cannot be factored into linear factors with rational coefficients. The next example shows how to find the zeros associated with *reducible* quadratics. In this example, factoring is used to find the zeros of each quadratic. Try using the Quadratic Formula to obtain the same zeros.

EXAMPLE 2 Factoring Quadratics

Find the zeros of each quadratic polynomial.

(a) $x^2 - 5x + 6$ (b) $x^2 - 6x + 9$ (c) $2x^2 + 5x - 3$

SOLUTION

(a) Because

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

the zeros are $x = 2$ and $x = 3$.

(b) Because

$$x^2 - 6x + 9 = (x - 3)^2$$

the only zero is $x = 3$.

(c) Because

$$2x^2 + 5x - 3 = (2x - 1)(x + 3)$$

the zeros are $x = \frac{1}{2}$ and $x = -3$.

ALGEBRA REVIEW

The zeros of a polynomial in x are the values of x that make the polynomial zero. To find the zeros, factor the polynomial into linear factors and set each factor equal to zero. For instance, the zeros of $(x - 2)(x - 3)$ occur when $x - 2 = 0$ and $x - 3 = 0$.

TRY IT 2

Find the zeros of each quadratic polynomial.

(a) $x^2 - 2x - 15$ (b) $x^2 + 2x + 1$ (c) $2x^2 - 7x + 6$

EXAMPLE 3 Finding the Domain of a Radical Expression

Find the domain of $\sqrt{x^2 - 3x + 2}$.

SOLUTION Because

$$x^2 - 3x + 2 = (x - 1)(x - 2)$$

you know that the zeros of the quadratic are $x = 1$ and $x = 2$. So, you need to test the sign of the quadratic in the three intervals $(-\infty, 1)$, $(1, 2)$, and $(2, \infty)$, as shown in Figure 0.15. After testing each of these intervals, you can see that the quadratic is negative in the center interval and positive in the outer two intervals. Moreover, because the quadratic is zero when $x = 1$ and $x = 2$, you can conclude that the domain of $\sqrt{x^2 - 3x + 2}$ is

$$(-\infty, 1] \cup [2, \infty).$$ Domain

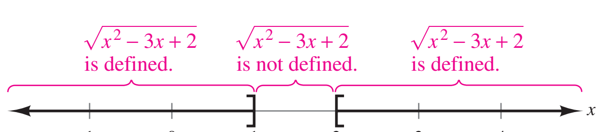


FIGURE 0.15

Values of $\sqrt{x^2 - 3x + 2}$

x	$\sqrt{x^2 - 3x + 2}$
0	$\sqrt{2}$
1	0
1.5	Undefined
2	0
3	$\sqrt{2}$

TRY IT 3

Find the domain of

$$\sqrt{x^2 + x - 2}.$$

Factoring Polynomials of Degree Three or More

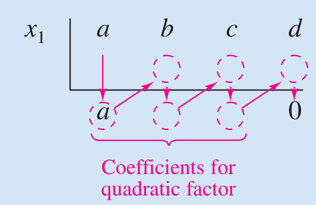
It can be difficult to find the zeros of polynomials of degree three or more. However, if one of the zeros of a polynomial is known, then you can use that zero to reduce the degree of the polynomial. For example, if you know that $x = 2$ is a zero of $x^3 - 4x^2 + 5x - 2$, then you know that $(x - 2)$ is a factor, and you can use long division to factor the polynomial as shown.

$$\begin{aligned} x^3 - 4x^2 + 5x - 2 &= (x - 2)(x^2 - 2x + 1) \\ &= (x - 2)(x - 1)(x - 1) \end{aligned}$$

As an alternative to long division, many people prefer to use **synthetic division** to reduce the degree of a polynomial.

Synthetic Division for a Cubic Polynomial

Given: $x = x_1$ is a zero of $ax^3 + bx^2 + cx + d$.



Vertical pattern: *Add terms.*

Diagonal pattern: *Multiply by* x_1.

Performing synthetic division on the polynomial

$$x^3 - 4x^2 + 5x - 2$$

using the given zero, $x = 2$, produces the following.

$$\begin{array}{r|rrrr} 2 & 1 & -4 & 5 & -2 \\ & & 2 & -4 & 2 \\ \hline & 1 & -2 & 1 & 0 \end{array}$$

$(x - 2)(x^2 - 2x + 1) = x^3 - 4x^2 + 5x - 2$

When you use synthetic division, remember to take *all* coefficients into account—*even if some of them are zero*. For instance, if you know that $x = -2$ is a zero of

$$x^3 + 3x + 14$$

you can apply synthetic division as shown.

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 3 & 14 \\ & & -2 & 4 & -14 \\ \hline & 1 & -2 & 7 & 0 \end{array}$$

$(x + 2)(x^2 - 2x + 7) = x^3 + 3x + 14$

STUDY TIP

Note that synthetic division works *only* for divisors of the form $x - x_1$. [Remember that $x + x_1 = x - (-x_1)$.] You cannot use synthetic division to divide a polynomial by a quadratic such as $x^2 - 3$.

The Rational Zero Theorem

There is a systematic way to find the *rational* zeros of a polynomial. You can use the **Rational Zero Theorem** (also called the Rational Root Theorem).

Rational Zero Theorem

If a polynomial

$$a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1x + a_0$$

has integer coefficients, then every *rational* zero is of the form $x = p/q$, where p is a factor of a_0, and q is a factor of a_n.

EXAMPLE 4 Using the Rational Zero Theorem

Find all real zeros of the polynomial.

$$2x^3 + 3x^2 - 8x + 3$$

SOLUTION

$$2x^3 + 3x^2 - 8x + 3$$

Factors of constant term: $\pm 1, \pm 3$

Factors of leading coefficient: $\pm 1, \pm 2$

The possible rational zeros are the factors of the constant term divided by the factors of the leading coefficient.

$$1, -1, 3, -3, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}$$

By testing these possible zeros, you can see that $x = 1$ works.

$$2(1)^3 + 3(1)^2 - 8(1) + 3 = 2 + 3 - 8 + 3 = 0$$

Now, by synthetic division you have the following.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

$$(x - 1)(2x^2 + 5x - 3) = 2x^3 + 3x^2 - 8x + 3$$

Finally, by factoring the quadratic, $2x^2 + 5x - 3 = (2x - 1)(x + 3)$, you have

$$2x^3 + 3x^2 - 8x + 3 = (x - 1)(2x - 1)(x + 3)$$

and you can conclude that the zeros are $x = 1$, $x = \frac{1}{2}$, and $x = -3$.

STUDY TIP

In Example 4, you can check that the zeros are correct by substituting into the original polynomial.

Check that $x = 1$ is a zero.

$$2(1)^3 + 3(1)^2 - 8(1) + 3$$
$$= 2 + 3 - 8 + 3$$
$$= 0$$

Check that $x = \frac{1}{2}$ is a zero.

$$2\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 - 8\left(\frac{1}{2}\right) + 3$$
$$= \frac{1}{4} + \frac{3}{4} - 4 + 3$$
$$= 0$$

Check that $x = -3$ is a zero.

$$2(-3)^3 + 3(-3)^2 - 8(-3) + 3$$
$$= -54 + 27 + 24 + 3$$
$$= 0$$

TRY IT 4

Find all real zeros of the polynomial.

$$2x^3 - 3x^2 - 3x + 2$$

EXERCISES 0.4

In Exercises 1–8, use the Quadratic Formula to find all real zeros of the second-degree polynomial.

1. $6x^2 - x - 1$
2. $8x^2 - 2x - 1$
3. $4x^2 - 12x + 9$
4. $9x^2 + 12x + 4$
5. $y^2 + 4y + 1$
6. $x^2 + 6x - 1$
7. $2x^2 + 3x - 4$
8. $3x^2 - 8x - 4$

In Exercises 9–18, write the second-degree polynomial as the product of two linear factors.

9. $x^2 - 4x + 4$
10. $x^2 + 10x + 25$
11. $4x^2 + 4x + 1$
12. $9x^2 - 12x + 4$
13. $x^2 + x - 2$
14. $2x^2 - x - 1$
15. $3x^2 - 5x + 2$
16. $x^2 - xy - 2y^2$
17. $x^2 - 4xy + 4y^2$
18. $a^2b^2 - 2abc + c^2$

In Exercises 19–34, completely factor the polynomial.

19. $81 - y^4$
20. $x^4 - 16$
21. $x^3 - 8$
22. $y^3 - 64$
23. $y^3 + 64$
24. $z^3 + 125$
25. $x^3 - 27$
26. $(x - a)^3 + b^3$
27. $x^3 - 4x^2 - x + 4$
28. $x^3 - x^2 - x + 1$
29. $2x^3 - 3x^2 + 4x - 6$
30. $x^3 - 5x^2 - 5x + 25$
31. $2x^3 - 4x^2 - x + 2$
32. $x^3 - 7x^2 - 4x + 28$
33. $x^4 - 15x^2 - 16$
34. $2x^4 - 49x^2 - 25$

In Exercises 35–52, find all real zeros of the polynomial.

35. $x^2 - 5x$
36. $2x^2 - 3x$
37. $x^2 - 9$
38. $x^2 - 25$
39. $x^2 - 3$
40. $x^2 - 8$
41. $(x - 3)^2 - 9$
42. $(x + 1)^2 - 8$
43. $x^2 + x - 2$
44. $x^2 + 5x + 6$
45. $x^2 - 5x + 6$
46. $x^2 + x - 20$
47. $x^3 + 64$
48. $x^3 - 216$
49. $x^4 - 16$
50. $x^4 - 625$
51. $x^3 - x^2 - 4x + 4$
52. $2x^3 + x^2 + 6x + 3$

In Exercises 53–56, find the interval (or intervals) on which the given expression is defined.

53. $\sqrt{x^2 - 4}$
54. $\sqrt{4 - x^2}$
55. $\sqrt{x^2 - 7x + 12}$
56. $\sqrt{x^2 - 8x + 15}$

In Exercises 57–60, use synthetic division to complete the indicated factorization.

57. $x^3 - 3x^2 - 6x - 2 = (x + 1)(\quad)$
58. $x^3 - 2x^2 - x + 2 = (x + 1)(\quad)$
59. $2x^3 - x^2 - 2x + 1 = (x - 1)(\quad)$
60. $x^4 - 16x^3 + 96x^2 - 256x + 256 = (x - 4)(\quad)$

In Exercises 61–68, use the Rational Zero Theorem as an aid in finding all real zeros of the polynomial.

61. $x^3 - x^2 - 10x - 8$
62. $x^3 - 7x - 6$
63. $x^3 - 6x^2 + 11x - 6$
64. $x^3 + 2x^2 - 5x - 6$
65. $6x^3 - 11x^2 - 19x - 6$
66. $18x^3 - 9x^2 - 8x + 4$
67. $x^3 - 3x^2 - 3x - 4$
68. $2x^3 - x^2 - 13x - 6$

69. ***Average Cost*** The minimum average cost of producing x units of a product occurs when the production level is set at the (positive) solution of

$$0.0003x^2 - 1200 = 0.$$

Determine this production level.

70. ***Profit*** The profit P from sales is given by

$$P = -200x^2 + 2000x - 3800$$

where x is the number of units sold per day (in hundreds). Determine the interval for x such that the profit will be greater than 1000.

71. ***Chemistry: Finding Concentrations*** Use the Quadratic Formula to solve the expression

$$1.8 \times 10^{-5} = \frac{x^2}{1.0 \times 10^{-4} - x}$$

which is needed to determine the quantity of hydrogen ions ($[H^+]$) in a solution of 1.0×10^{-4}M acetic acid. Because x represents a concentration of $[H^+]$, only positive values of x are possible solutions. *(Source: Adapted from Zumdahl,* Chemistry, *Sixth Edition)*

72. ***Finance*** After 2 years, an investment of \$1200 is made at an interest rate r, compounded annually, that will yield an amount of

$$A = 1200(1 + r)^2.$$

Determine the interest rate if $A = \$1300$.