Continuity

In mathematics, the term “continuous” has much the same meaning as it does in everyday use. To say that a function is continuous at \( x = c \) means that there is no interruption in the graph of \( f \) at \( c \). The graph of \( f \) is unbroken at \( c \) and there are no holes, jumps, or gaps. As simple as this concept may seem, its precise definition eluded mathematicians for many years. In fact, it was not until the early 1800s that a precise definition was finally developed.

Before looking at this definition, consider the function whose graph is shown in Figure 1.60. This figure identifies three values of \( x \) at which the function \( f \) is not continuous.
1. At \( x = c_1 \), \( f(c_1) \) is not defined.
2. At \( x = c_2 \), \( \lim_{x \to c_2} f(x) \) does not exist.
3. At \( x = c_3 \), \( f(c_3) \neq \lim_{x \to c_3} f(x) \).

At all other points in the interval \((a, b)\), the graph of \( f \) is uninterrupted, which implies that the function \( f \) is continuous at all other points in the interval \((a, b)\).

Definition of Continuity

Let \( c \) be a number in the interval \((a, b)\), and let \( f \) be a function whose domain contains the interval \((a, b)\). The function \( f \) is **continuous at the point** \( c \) if the following conditions are true.
1. \( f(c) \) is defined.
2. \( \lim_{x \to c} f(x) \) exists.
3. \( \lim_{x \to c} f(x) = f(c) \).

If \( f \) is continuous at every point in the interval \((a, b)\), then it is **continuous on an open interval** \((a, b)\).

Roughly, you can say that a function is continuous on an interval if its graph on the interval can be traced using a pencil and paper without lifting the pencil from the paper, as shown in Figure 1.61.
In Section 1.5, you studied several types of functions that meet the three conditions for continuity. Specifically, if direct substitution can be used to evaluate the limit of a function at \( c \), then the function is continuous at \( c \). Two types of functions that have this property are polynomial functions and rational functions.

### Continuity of Polynomial and Rational Functions

1. A polynomial function is continuous at every real number.
2. A rational function is continuous at every number in its domain.

#### Example 1  Determining Continuity of a Polynomial Function

Discuss the continuity of each function.

(a) \( f(x) = x^2 - 2x + 3 \)

(b) \( f(x) = x^3 - x \)

**Solution** Each of these functions is a polynomial function. So, each is continuous on the entire real line, as indicated in Figure 1.62.

![Graphs of polynomial functions](image)

**Figure 1.62** Both functions are continuous on \((-\infty, \infty)\).

#### Try It 1

Discuss the continuity of each function.

(a) \( f(x) = x^2 + x + 1 \)  
(b) \( f(x) = x^3 + x \)

Polynomial functions are one of the most important types of functions used in calculus. Be sure you see from Example 1 that the graph of a polynomial function is continuous on the entire real line, and therefore has no holes, jumps, or gaps. Rational functions, on the other hand, need not be continuous on the entire real line, as shown in Example 2.
Consider an open interval $I$ that contains a real number $c$. If a function $f$ is defined on $I$ (except possibly at $c$), and $f$ is not continuous at $c$, then $f$ is said to have a discontinuity at $c$. Discontinuities fall into two categories: removable and nonremovable. A discontinuity at $c$ is called removable if $f$ can be made continuous by appropriately defining (or redefining) $f(c)$. For instance, the function in Example 2(b) has a removable discontinuity at $(1, 2)$. To remove the discontinuity, all you need to do is redefine the function so that $f(1) = 2$.

A discontinuity at $x = c$ is nonremovable if the function cannot be made continuous at $x = c$ by defining or redefining the function at $x = c$. For instance, the function in Example 2(a) has a nonremovable discontinuity at $x = 0$. 

**EXAMPLE 2**

**Determining Continuity of a Rational Function**

Discuss the continuity of each function.

(a) $f(x) = \frac{1}{x}$  
(b) $f(x) = \frac{x^2 - 1}{x - 1}$  
(c) $f(x) = \frac{1}{x^2 + 1}$

**SOLUTION**

Each of these functions is a rational function and is therefore continuous at every number in its domain.

(a) The domain of $f(x) = \frac{1}{x}$ consists of all real numbers except $x = 0$. So, this function is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$. [See Figure 1.63(a).]

(b) The domain of $f(x) = \frac{x^2 - 1}{x - 1}$ consists of all real numbers except $x = 1$. So, this function is continuous on the intervals $(-\infty, 1)$ and $(1, \infty)$. [See Figure 1.63(b).]

(c) The domain of $f(x) = \frac{1}{x^2 + 1}$ consists of all real numbers. So, this function is continuous on the entire real line. [See Figure 1.63(c).]

**FIGURE 1.63**

(a) Continuous on $(-\infty, 0)$ and $(0, \infty)$  
(b) Continuous on $(-\infty, 1)$ and $(1, \infty)$  
(c) Continuous on $(-\infty, \infty)$

**TRY IT 2**

Discuss the continuity of each function.

(a) $f(x) = \frac{1}{x - 1}$  
(b) $f(x) = \frac{x^2 - 4}{x - 2}$  
(c) $f(x) = \frac{1}{x^2 + 2}$

Consider an open interval $I$ that contains a real number $c$. If a function $f$ is defined on $I$ (except possibly at $c$), and $f$ is not continuous at $c$, then $f$ is said to have a discontinuity at $c$. Discontinuities fall into two categories: removable and nonremovable. A discontinuity at $c$ is called removable if $f$ can be made continuous by appropriately defining (or redefining) $f(c)$. For instance, the function in Example 2(b) has a removable discontinuity at $(1, 2)$. To remove the discontinuity, all you need to do is redefine the function so that $f(1) = 2$.

A discontinuity at $x = c$ is nonremovable if the function cannot be made continuous at $x = c$ by defining or redefining the function at $x = c$. For instance, the function in Example 2(a) has a nonremovable discontinuity at $x = 0$. 

**SECTION 1.6 Continuity**
Continuity on a Closed Interval

The intervals discussed in Examples 1 and 2 are open. To discuss continuity on a closed interval, you can use the concept of one-sided limits, as defined in Section 1.5.

Definition of Continuity on a Closed Interval

Let $f$ be defined on a closed interval $[a, b]$. If $f$ is continuous on the open interval $(a, b)$ and

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b)$$

then $f$ is continuous on the closed interval $[a, b]$. Moreover, $f$ is continuous from the right at $a$ and continuous from the left at $b$.

Similar definitions can be made to cover continuity on intervals of the form $(a, b]$ and $[a, b)$, or on infinite intervals. For example, the function

$$f(x) = \sqrt{x}$$

is continuous on the infinite interval $[0, \infty)$.

Example 3: Examining Continuity at an Endpoint

Discuss the continuity of

$$f(x) = \sqrt{3 - x}.$$  

Solution: Notice that the domain of $f$ is the set $(-\infty, 3]$. Moreover, $f$ is continuous from the left at $x = 3$ because

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{3 - x} = 0 = f(3).$$

For all $x < 3$, the function $f$ satisfies the three conditions for continuity. So, you can conclude that $f$ is continuous on the interval $(-\infty, 3]$, as shown in Figure 1.64.

Try It 3

Discuss the continuity of $f(x) = \sqrt{x - 2}$.

Study Tip

When working with radical functions of the form

$$f(x) = \sqrt{g(x)}$$

remember that the domain of $f$ coincides with the solution of $g(x) \geq 0$. 
The Greatest Integer Function

Many functions that are used in business applications are step functions. For instance, the function in Example 9 in Section 1.5 is a step function. The greatest integer function is another example of a step function. This function is denoted by

\[
[x] = \text{greatest integer less than or equal to } x.
\]

For example,

\[
[-2.1] = \text{greatest integer less than or equal to } -2.1 = -3 \\
[-2] = \text{greatest integer less than or equal to } -2 = -2 \\
[1.5] = \text{greatest integer less than or equal to } 1.5 = 1.
\]

Note that the graph of the greatest integer function (Figure 1.66) jumps up one unit at each integer. This implies that the function is not continuous at each integer.

In real-life applications, the domain of the greatest integer function is often restricted to nonnegative values of \(x\). In such cases this function serves the purpose of truncating the decimal portion of \(x\). For example, 1.345 is truncated to 1 and 3.57 is truncated to 3. That is,

\[
[1.345] = 1 \quad \text{and} \quad [3.57] = 3.
\]
A bookbinding company produces 10,000 books in an eight-hour shift. The fixed cost per shift amounts to $5000, and the unit cost per book is $3. Using the greatest integer function, you can write the cost of producing \( x \) books as

\[
C = 5000\left(1 + \left\lfloor \frac{x - 1}{10,000} \right\rfloor \right) + 3x.
\]

Sketch the graph of this cost function.

**SOLUTION** Note that during the first eight-hour shift

\[
\left\lfloor \frac{x - 1}{10,000} \right\rfloor = 0, \quad 1 \leq x \leq 10,000
\]

which implies

\[
C = 5000\left(1 + \left\lfloor \frac{x - 1}{10,000} \right\rfloor \right) + 3x = 5000 + 3x.
\]

During the second eight-hour shift

\[
\left\lfloor \frac{x - 1}{10,000} \right\rfloor = 1, \quad 10,001 \leq x \leq 20,000
\]

which implies

\[
C = 5000\left(1 + \left\lfloor \frac{x - 1}{10,000} \right\rfloor \right) + 3x = 10,000 + 3x.
\]

The graph of \( C \) is shown in Figure 1.67. Note the graph’s discontinuities.

**TRY IT 5** Use a graphing utility to graph the cost function in Example 5.
To graph a step function or compound function with a graphing utility, you must be familiar with the utility’s programming language. For instance, different graphing utilities have different “integer truncation” functions. One is $\text{IPart}(x)$, and it yields the truncated integer part of $x$. For example, $\text{IPart}(-1.2) = -1$ and $\text{IPart}(3.4) = 3$. The other function is $\text{Int}(x)$, which is the greatest integer function. The graphs of these two functions are shown below. When graphing a step function, you should set your graphing utility to *dot mode*.

On some graphing utilities, you can graph a piecewise-defined function such as

$$f(x) = \begin{cases} 
  x^2 - 4, & x \leq 2 \\
  -x + 2, & 2 < x 
\end{cases}$$

The graph of this function is shown below.

Consult the user’s guide for your graphing utility for specific keystrokes you can use to graph these functions.
Extended Application: Compound Interest

Banks and other financial institutions differ on how interest is paid to an account. If the interest is added to the account so that future interest is paid on previously earned interest, then the interest is said to be compounded. Suppose, for example, that you deposited $10,000 in an account that pays 6% interest, compounded quarterly. Because the 6% is the annual interest rate, the quarterly rate is \( \frac{0.06}{4} = 0.015 \). The balances during the first five quarters are shown below.

<table>
<thead>
<tr>
<th>Quarter</th>
<th>Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>$10,000.00</td>
</tr>
<tr>
<td>2nd</td>
<td>10,000.00 + (0.015)(10,000.00) = $10,100.00</td>
</tr>
<tr>
<td>3rd</td>
<td>10,100.00 + (0.015)(10,100.00) = $10,200.00</td>
</tr>
<tr>
<td>4th</td>
<td>10,200.00 + (0.015)(10,200.00) = $10,300.00</td>
</tr>
<tr>
<td>5th</td>
<td>10,300.00 + (0.015)(10,300.00) = $10,400.00</td>
</tr>
</tbody>
</table>

### TECHNOLOGY

You can use a spreadsheet or the table feature of a graphing utility to create a table. Try doing this for the data shown at the right. (Consult the user’s manual of a spreadsheet software program for specific instructions on how to create a table.)

### EXAMPLE 6  Graphing Compound Interest

Sketch the graph of the balance in the account described above.

**SOLUTION**  
Let \( A \) represent the balance in the account and let \( t \) represent the time, in years. You can use the greatest integer function to represent the balance, as shown.

\[
A = 10,000(1 + 0.015)^{\lfloor 4t \rfloor}
\]

From the graph shown in Figure 1.68, notice that the function has a discontinuity at each quarter.

### TRY IT 6

Write an equation that gives the balance of the account in Example 6 if the annual interest rate is 8%.

### TAKE ANOTHER LOOK

**Compound Interest**

If \( P \) dollars is deposited in an account, compounded \( n \) times per year, with an annual rate of \( r \) (in decimal form), then the balance \( A \) after \( t \) years is given by

\[
A = P \left(1 + \frac{r}{n}\right)^{nt}.
\]

Sketch the graph of each function. Which function is continuous? Describe the differences in policy between a bank that uses the first formula and a bank that uses the second formula.

a. \( A = P \left(1 + \frac{r}{n}\right)^t \)  
   b. \( A = P \left(1 + \frac{r}{n}\right)^{\lfloor nt \rfloor} \)
In Exercises 1–4, simplify the expression.

1. \( \frac{x^2 + 6x + 8}{x^2 - 6x - 16} \)
2. \( \frac{x^2 - 5x - 6}{x^2 - 9x + 18} \)
3. \( \frac{2x^2 - 2x - 12}{4x^2 - 24x + 36} \)
4. \( \frac{x^3 - 16x}{x^3 + 2x^2 - 8x} \)

In Exercises 5–8, solve for \( x \).

5. \( x^2 + 7x = 0 \)
6. \( x^2 + 4x - 5 = 0 \)
7. \( 3x^2 + 8x + 4 = 0 \)
8. \( x^3 + 5x^2 - 24x = 0 \)

In Exercises 9 and 10, find the limit.

9. \( \lim_{x \to 3} (2x^2 - 3x + 4) \)
10. \( \lim_{x \to -2} (3x^3 - 8x + 7) \)

In Exercises 11–14, determine whether the function is continuous on the entire real line. Explain your reasoning.

11. \( f(x) = 5x^3 - x^2 + 2 \)
12. \( f(x) = (x^2 - 1)^3 \)
13. \( f(x) = \frac{x^2 - 1}{x + 1} \)
14. \( f(x) = \frac{x^3 - 8}{x - 2} \)
15. \( f(x) = x^3 - 2x + 1 \)
16. \( f(x) = 3 - 2x - x^3 \)
17. \( f(x) = \frac{x}{x^2 - 1} \)
18. \( f(x) = \frac{x - 3}{x^2 - 9} \)
19. \( f(x) = \frac{x}{x^2 + 1} \)
20. \( f(x) = \frac{1}{x^2 + 1} \)
21. \( f(x) = \frac{x - 5}{x^2 - 9x + 20} \)
22. \( f(x) = \frac{x - 1}{x^2 + x - 2} \)

In Exercises 11–34, describe the interval(s) on which the function is continuous.

11. \( f(x) = \frac{x^2 - 1}{x} \)
12. \( f(x) = \frac{1}{x^2 - 4} \)
23. \( f(x) = [2x] + 1 \)

24. \( f(x) = \frac{|x|}{2} + x \)

25. \( f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \geq 1 \end{cases} \)

26. \( f(x) = \begin{cases} 3 + x, & x \leq 2 \\ x^2 + 1, & x > 2 \end{cases} \)

27. \( f(x) = \begin{cases} \frac{1}{x} + 1, & x \leq 2 \\ 3 - x, & x > 2 \end{cases} \)

28. \( f(x) = \begin{cases} x^2 - 4, & x \leq 0 \\ 3x + 1, & x > 0 \end{cases} \)

29. \( f(x) = \frac{|x + 1|}{x + 1} \)

30. \( f(x) = \frac{4 - x}{4 - x} \)

31. \( f(x) = \|x - 1\| \)

32. \( f(x) = x - |x| \)

33. \( h(x) = f(g(x)), \quad f(x) = \frac{1}{\sqrt{x}}, \quad g(x) = x - 1, x > 1 \)

34. \( h(x) = f(g(x)), \quad f(x) = \frac{1}{x - 1}, \quad g(x) = x^2 + 5 \)

In Exercises 35–38, discuss the continuity of the function on the closed interval. If there are any discontinuities, determine whether they are removable.

<table>
<thead>
<tr>
<th>Function</th>
<th>Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>35. ( f(x) = x^2 - 4x - 5 )</td>
<td>([-1, 5])</td>
</tr>
<tr>
<td>36. ( f(x) = \frac{5}{x^2 + 1} )</td>
<td>([-2, 5])</td>
</tr>
<tr>
<td>37. ( f(x) = \frac{1}{x - 2} )</td>
<td>([1, 2])</td>
</tr>
<tr>
<td>38. ( f(x) = \frac{x}{x^2 - 4x + 3} )</td>
<td>([0, 4])</td>
</tr>
</tbody>
</table>

In Exercises 39–44, sketch the graph of the function and describe the interval(s) on which the function is continuous.

39. \( f(x) = \frac{x^2 - 16}{x - 4} \)

40. \( f(x) = \frac{2x^2 + x}{x} \)

41. \( f(x) = \frac{x^3 + x}{x} \)

42. \( f(x) = \frac{x - 3}{4x^3 - 12x} \)

43. \( f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x - 1, & x \geq 0 \end{cases} \)

44. \( f(x) = \begin{cases} x^2 - 4, & x \leq 0 \\ 2x + 4, & x > 0 \end{cases} \)

In Exercises 45 and 46, find the constant \( a \) (Exercise 45) and the constants \( a \) and \( b \) (Exercise 46) such that the function is continuous on the entire real line.

45. \( f(x) = \begin{cases} x^3, & x \leq 2 \\ ax^2, & x > 2 \end{cases} \)

46. \( f(x) = \begin{cases} 2, & x \leq -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \geq 3 \end{cases} \)

In Exercises 47–52, use a graphing utility to graph the function. Use the graph to determine any \( x \)-values at which the function is not continuous.

47. \( h(x) = \frac{1}{x^2 - x - 2} \)

48. \( k(x) = \frac{x - 4}{x^3 - 5x + 4} \)

49. \( f(x) = \begin{cases} 2x - 4, & x \leq 3 \\ x^2 - 2x, & x > 3 \end{cases} \)

50. \( f(x) = \begin{cases} 3x - 1, & x \leq 1 \\ x + 1, & x > 1 \end{cases} \)

51. \( f(x) = x - 2|x| \)

52. \( f(x) = |2x - 1| \)

In Exercises 53–56, describe the interval(s) on which the function is continuous.

53. \( f(x) = \frac{x}{x^2 + 1} \)

54. \( f(x) = x\sqrt{x + 3} \)
55. \( f(x) = \frac{1}{2}[2x] \)

56. \( f(x) = \frac{x + 1}{\sqrt{x}} \)

59. **Compound Interest**  A deposit of $7500 is made in an account that pays 6% compounded quarterly. The amount \( A \) in the account after \( t \) years is

\[
A = 7500(1.015)^t, \quad t \geq 0.
\]

(a) Sketch the graph of \( A \). Is the graph continuous? Explain your reasoning.

(b) What is the balance after 7 years?

60. **Environmental Cost**  The cost \( C \) (in millions of dollars) of removing \( x \) percent of the pollutants emitted from the smokestack of a factory can be modeled by

\[
C = \frac{2 \times 100}{100 - x}.
\]

(a) What is the implied domain of \( C \)? Explain your reasoning.

(b) Use a graphing utility to graph the cost function. Is the function continuous on its domain? Explain your reasoning.

(c) Find the cost of removing 75% of the pollutants from the smokestack.

61. **Consumer Awareness**  A shipping company’s charge for sending an overnight package from New York to Atlanta is $9.80 for the first pound and $2.50 for each additional pound or fraction thereof. Use the greatest integer function to create a model for the charge \( C \) for overnight delivery of a package weighing \( x \) pounds. Use a graphing utility to graph the function, and discuss its continuity.

62. **Consumer Awareness**  A cab company charges $3 for the first mile and $0.25 for each additional mile or fraction thereof. Use the greatest integer function to create a model for the cost \( C \) of a cab ride \( n \) miles long. Use a graphing utility to graph the function, and discuss its continuity.

63. **Consumer Awareness**  A dial-direct long distance call between two cities costs $1.04 for the first 2 minutes and $0.36 for each additional minute or fraction thereof.

(a) Use the greatest integer function to write the cost \( C \) of a call in terms of the time \( t \) (in minutes). Sketch the graph of the cost function and discuss its continuity.

(b) Find the cost of a nine-minute call.

64. **Salary Contract**  A union contract guarantees a 9% yearly increase for 5 years. For a current salary of $28,500, the salary for the next 5 years is given by

\[
S = 28,500(1.09)^t, \quad t \geq 0.
\]

(a) Use the greatest integer function to write the salary \( S \) of a graphing utility to graph the salary function, and discuss its continuity.

(b) Find the salary during the fifth year (when \( t = 5 \)).

65. **Inventory Management**  The number of units in inventory in a small company is

\[
N = 25\left(2\left\lfloor \frac{t + 2}{2} \right\rfloor - t\right), \quad 0 \leq t \leq 12
\]

where the real number \( t \) is the time in months.

(a) Use the greatest integer function of a graphing utility to graph this function, and discuss its continuity.

(b) How often must the company replenish its inventory?

66. **Owning a Franchise**  You have purchased a franchise. You have determined a linear model for your revenue as a function of time. Is the model a continuous function? Would your actual revenue be a continuous function of time? Explain your reasoning.

67. **Biology**  The gestation period of rabbits is only 26 to 30 days. Therefore, the population of a form (rabbits’ home) can increase dramatically in a short period of time. The table gives the population of a form, where \( t \) is the time in months and \( N \) is the rabbit population.

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N )</td>
<td>2</td>
<td>8</td>
<td>10</td>
<td>14</td>
<td>10</td>
<td>15</td>
<td>12</td>
</tr>
</tbody>
</table>

Graph the population as a function of time. Find any points of discontinuity in the function. Explain your reasoning.