Chapter 3: Random Variables[1]

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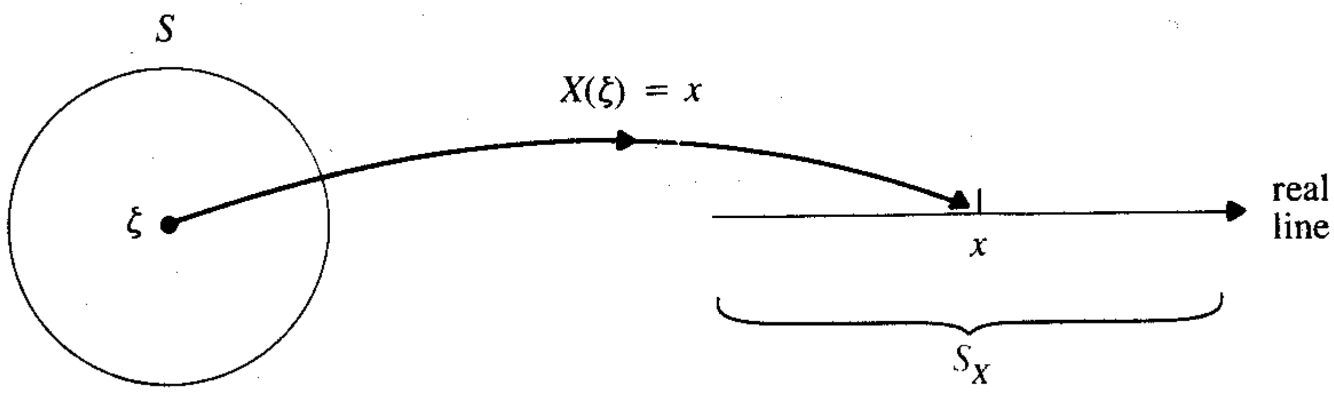
[1]Modified from the lecture notes by Prof. Mao-Ching Chiu

3.1 The Notion of a Random Variable

- A random variable X is a function that assigns a real number, $X(\zeta)$, to each outcome ζ in the sample space.



Example:

- Toss a coin three times.
- $$\begin{aligned} S &= \{\text{HHH}, \text{HHT}, \text{HTH}, \text{THH}, \text{HTT}, \\ &\quad \text{THT}, \text{TTH}, \text{TTT}\}. \end{aligned}$$
- Let X be the number of heads.

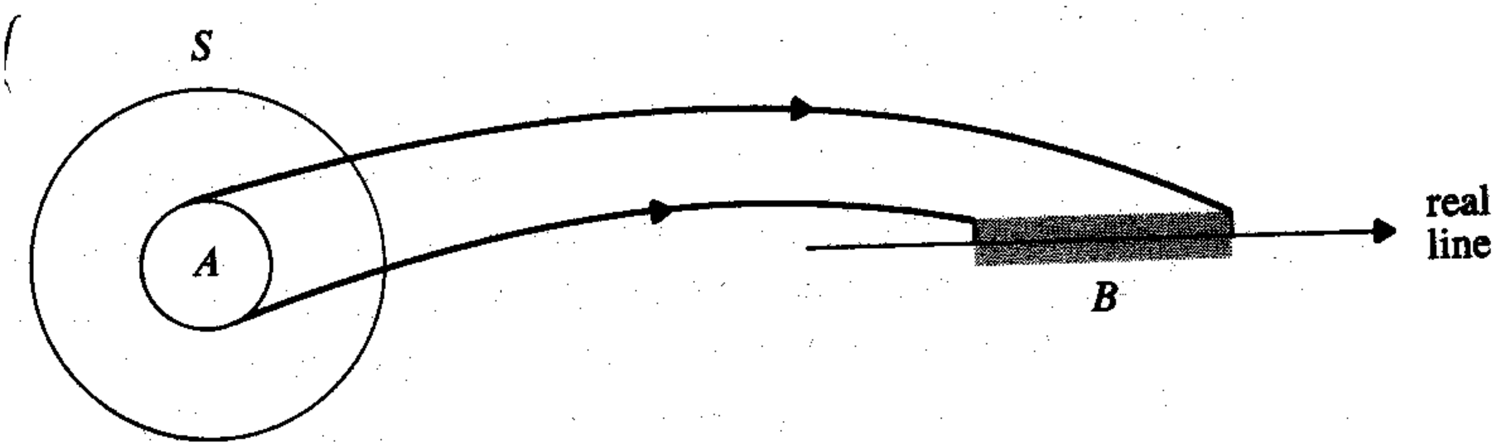
ζ	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
$X(\zeta)$	3	2	2	2	1	1	1	0

- X is a random variable with $S_X = \{0, 1, 2, 3\}$.

- S: Sample space of a random experiment
- X: Random variable $X : S \to S_X$
- S_X is a new sample space
- Let $B \subseteq S_X$ and $A = \{\zeta : X(\zeta) \in B\}$. Then

$$P[B] = P[A] = P[\{\zeta : X(\zeta) \in B\}]$$

- A and B are equivalent events.


S
A
B
real line

3.2 Cumulative Distribution Function

- Cumulative distribution function (cdf)

$$F_X(x) = P[X \leq x] \quad \text{for } -\infty < x < \infty$$

- In underlying sample space
  $F_X(x) = P[\{\zeta : X(\zeta) \leq x\}]$
- $F_X(x)$ is a function of the variable x.

Properties of cdf

1. $0 \leq F_X(x) \leq 1$.
2. $\lim_{x\to\infty} F_X(x) = 1$.
3. $\lim_{x\to-\infty} F_X(x) = 0$.
4. If $a < b$, then $F_X(a) \leq F_X(b)$.
5. $F_X(x)$ is continuous from the right, i.e., for $h > 0$
$$F_X(b) = \lim_{h\to 0} F_X(b+h) = F_X(b^+).$$
6. $P[a < X \leq b] = F_X(b) - F_X(a)$, since $\{X \leq a\} \cup \{a < X \leq b\} = \{X \leq b\}$.

7. $P[X = b] = F_X(b) - F_X(b^-)$.
8. $P[X > x] = 1 - F_X(x)$.

- X: the number of heads in three tosses of a fair coin
- Let δ be a small positive number. Then

$$F_X(1-\delta) = P[X \leq 1-\delta] = P\{0 \text{ heads}\} = 1/8,$$

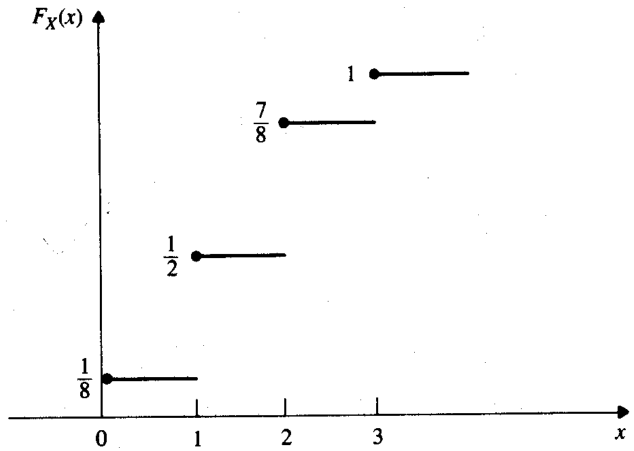
$$F_X(1) = P[X \leq 1] = P[0 \text{ or } 1 \text{ heads}] = 1/8+3/8 = 1/2,$$

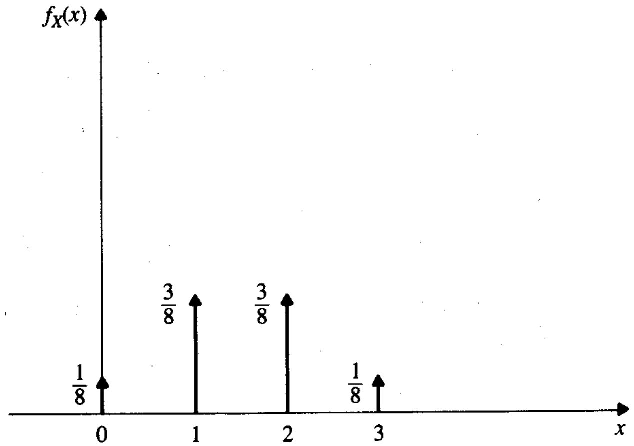
$$F_X(1+\delta) = P[X \leq 1+\delta] = P[0 \text{ or } 1 \text{ heads}] = 1/2.$$

- Write in unit step function

$$u(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0, \end{cases}$$

$$F_X(x) = \frac{1}{8}u(x) + \frac{3}{8}u(x-1) + \frac{3}{8}u(x-2) + \frac{1}{8}u(x-3).$$


$F_X(x)$
1
$\frac{7}{8}$
$\frac{1}{2}$
$\frac{1}{8}$
0
1
2
3
x
(a)


$f_X(x)$
$\frac{3}{8}$
$\frac{3}{8}$
$\frac{1}{8}$
$\frac{1}{8}$
0
1
2
3
x
(b)

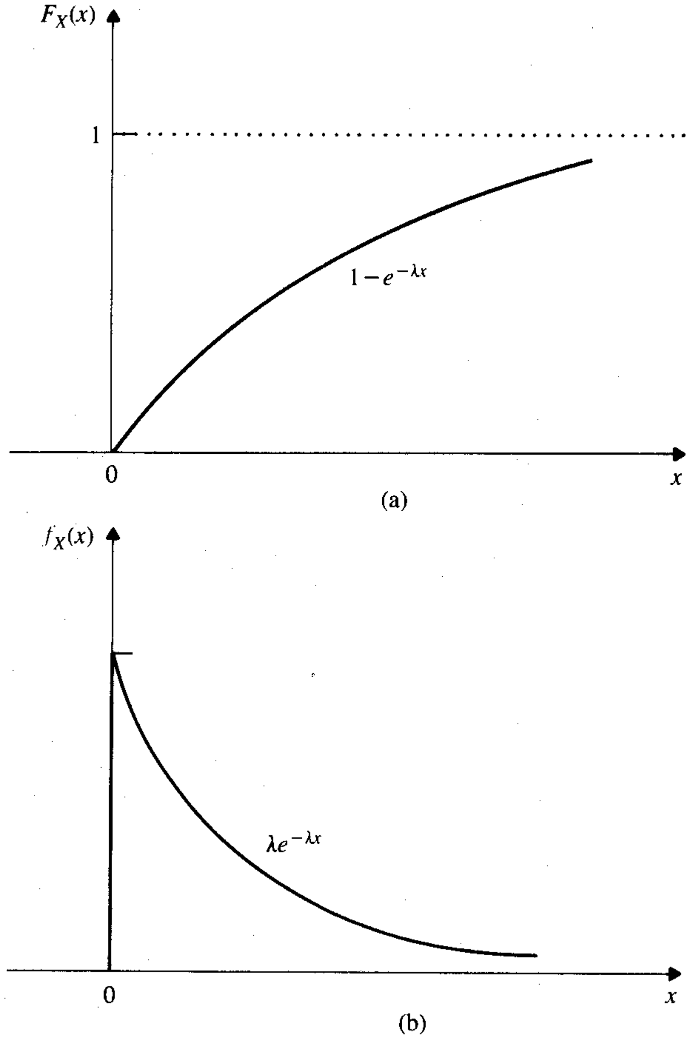
Example: The transmission time X of messages in a communication system obeys the exponential probability law with parameter λ, i.e.,

$$P[X > x] = e^{-\lambda x} \qquad x > 0$$

Find the cdf of X and $P[T < X \leq 2T]$, where $T = 1/\lambda$.

$$F_X(x) = P[X \leq x] = 1 - P[X > x] = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0. \end{cases}$$

$$P[T < X \leq 2T] = 1 - e^{-2} - (1 - e^{-1}) = e^{-1} - e^{-2}.$$



$F_X(x)$
1
$1-e^{-\lambda x}$
0
x
(a)
$f_X(x)$
$\lambda e^{-\lambda x}$
0
x
(b)

Types of random variables

- Discrete random variable, $S_X = x_0, x_1, \ldots$

$$F_X(x) = \sum_k p_X(x_k) u(x - x_k),$$

where $x_k \in S_X$ and $p_X(x_k) = P[X = x_k]$ is the probability mass function (pmf) of X.

- Continuous random variable

$$F_X(x) = \int_{-\infty}^{x} f(t) dt$$

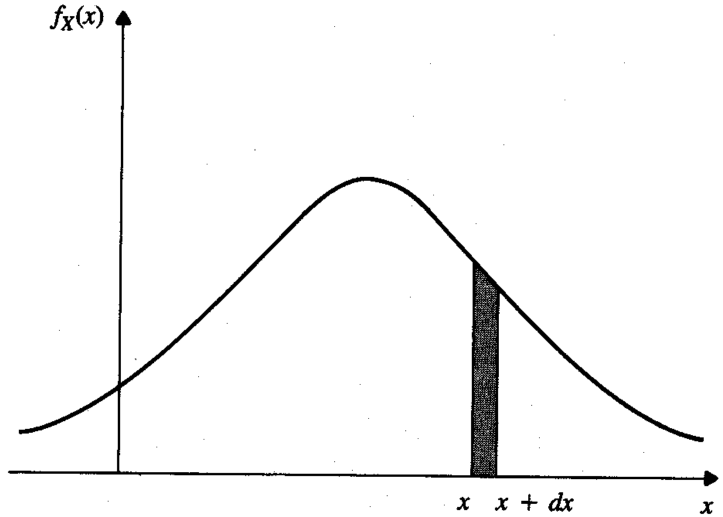
- Random variable of mixed type

$$F_X(x) = pF_1(x) + (1 - p)F_2(x)$$

3.3 Probability Density Function

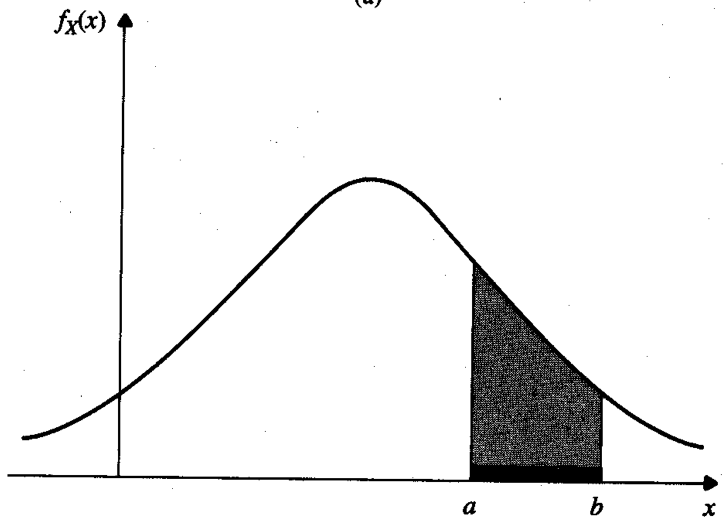
- Probability density function (pdf) of X, if it exists, is

$$f_X(x) = \frac{dF_X(x)}{dx}.$$



$P[x < X \leqslant x + dx] \cong f_X(x)dx$

(a)



$P[a \leqslant X \leqslant b] = \int_a^b f_X(x)dx$

(b)

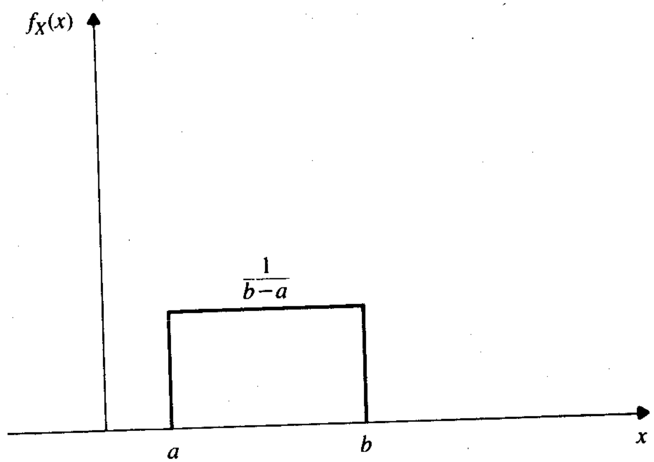
Properties of pdf

1. $f_X(x) \geq 0$ due to nondecreasing property of cdf.
2. $P[a \leq X \leq b] = \int_a^b f_X(x)dx$.
3. $F_X(x) = \int_{-\infty}^x f_X(t)dt$.
4. $\int_{-\infty}^{\infty} f_X(t)dt = 1$.

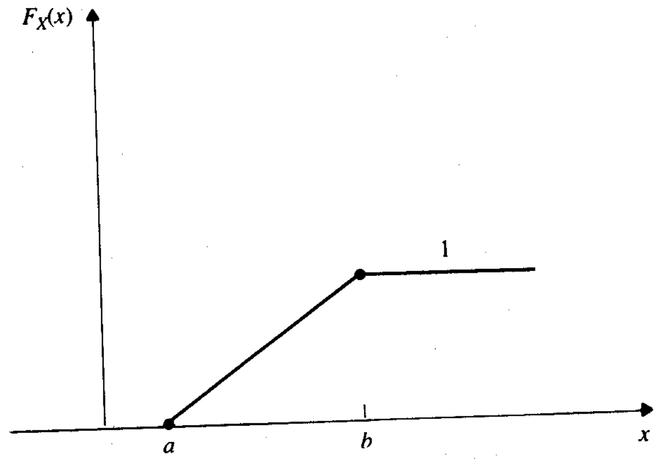
The pdf of the uniform random variable

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & otherwise \end{cases}$$

$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b \end{cases}$$



(a)



(b)

Pdf for discontinuous cdf

- Unit step function

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

- Delta function $\delta(t)$

$$u(x) = \int_{-\infty}^{x} \delta(t)dt$$

- Cdf for a discrete random variable

$$F_X(x) = \sum_k p_X(x_k)u(x - x_k) = \int_{-\infty}^{x} f_X(t)dt$$

$\rightarrow$ $f_X(x) = \sum_k p_X(x_k)\delta(x - x_k)$

Conditional cdf's and pdf's

- Conditional cdf of X given A

$$F_X(x|A) = \frac{P[\{X \le x\} \cap A]}{P[A]} \quad \text{if } P[A] > 0$$

- Conditional pdf of X given A

$$f_X(x|A) = \frac{d}{dx} F_X(x|A)$$

3.4 Some Important Random Variables

- **Bernoulli random variable**: Let A be an event. The indicator function for A is

$$I_A(\zeta) = \begin{cases} 0 & \zeta \notin A \\ 1 & \zeta \in A \end{cases}.$$

I_A is the Bernoulli random variable. Ex: toss a coin.

- **Binomial random variable**: Let X be the number of times a event A occurs in n independent trials. Let I_j be the indicator function for event A in the jth trial. Then

$$X = I_1 + I_2 + \cdots + I_n$$

and

$$P[X = k] = \binom{n}{k} p^k (1 - p)^{n-k},$$

where $p = P[I_j = 1]$.

- **Geometric random variable**: Count the number M of independent Bernoulli trials until the first success of event A.

$$P[M = k] = (1-p)^{k-1}p \qquad k = 1, 2, \ldots,$$

where $p = P[A]$.

- Another version of the geometric random variable is

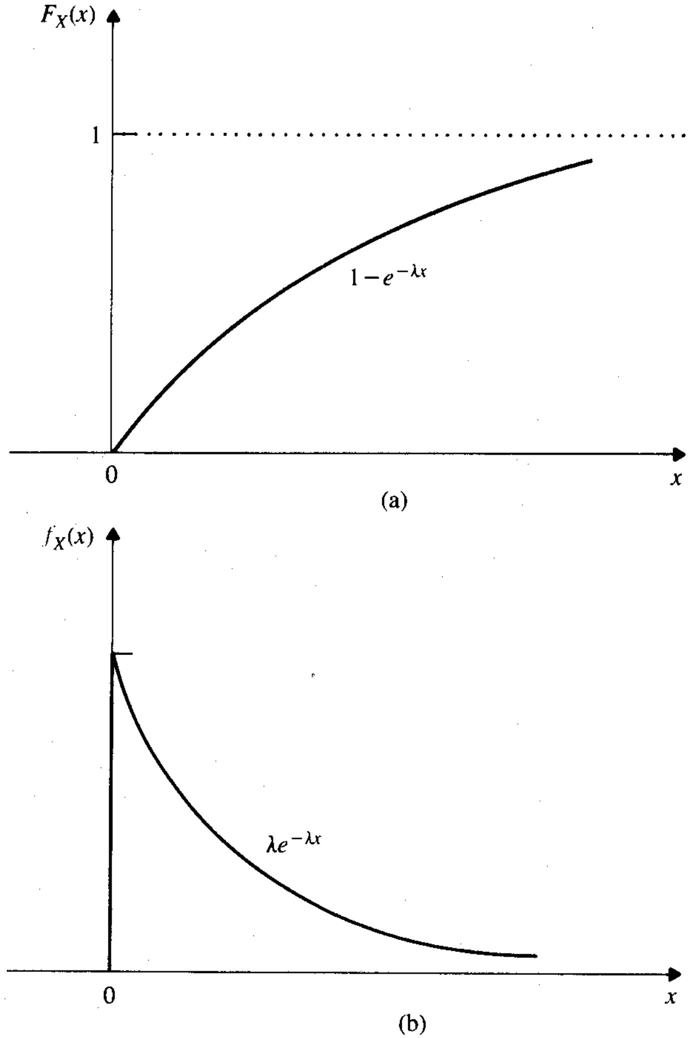
$$P[k] = (1-p)^{k}p \qquad k = 0, 1, 2, \ldots.$$

- **Exponential random variable**

$$f_X(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

λ: rate at which events occur.


$F_X(x)$
1
$1-e^{-\lambda x}$
0
x
(a)
$f_X(x)$
$\lambda e^{-\lambda x}$
0
x
(b)

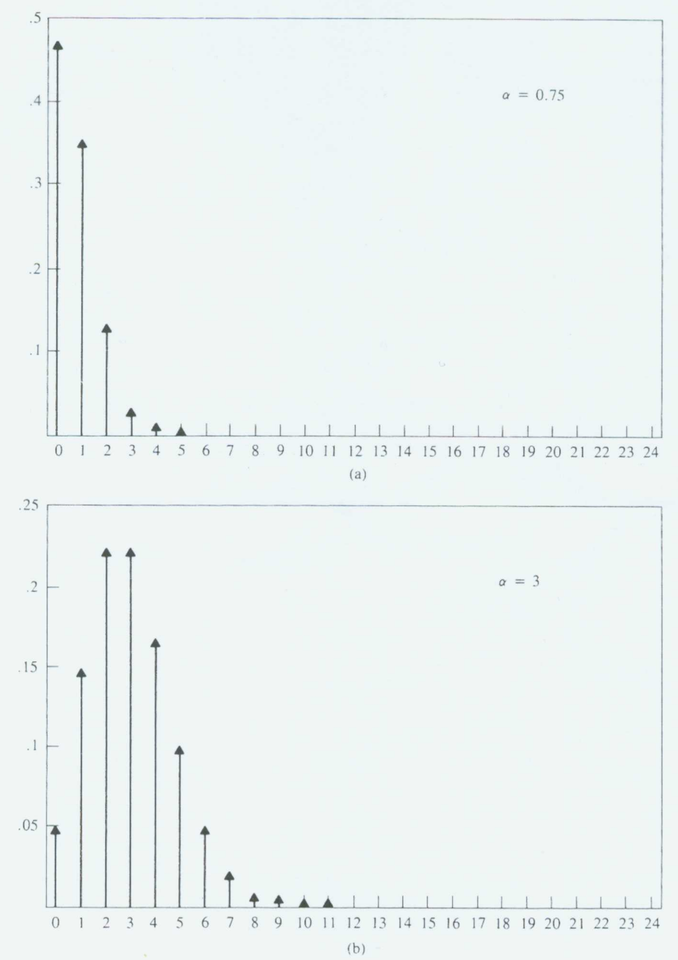
- **The Poisson random variable**:
  The pmf is

$$P[N = k] = \frac{\alpha^k}{k!}e^{-\alpha} \quad k = 0, 1, 2, , \ldots,$$

  where α is the average number of event occurrences in a specified time interval or region in space.

- The pmf of the Poison random variable sums to one, since

$$\sum_{k=0}^{\infty} \frac{\alpha^k}{k!}e^{-\alpha} = e^{-\alpha}\sum_{k=0}^{\infty} \frac{\alpha^k}{k!} = e^{-\alpha}e^{\alpha} = 1.$$


.5
.4
.3
.2
.1
α = 0.75
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
(a)
.25
.2
.15
.1
.05
α = 3
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
(b)

- **Gaussian (Normal) random variable**:

  The pdf is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2} \qquad -\infty < x < \infty,$$

  where m and σ are real numbers.

  The cdf is

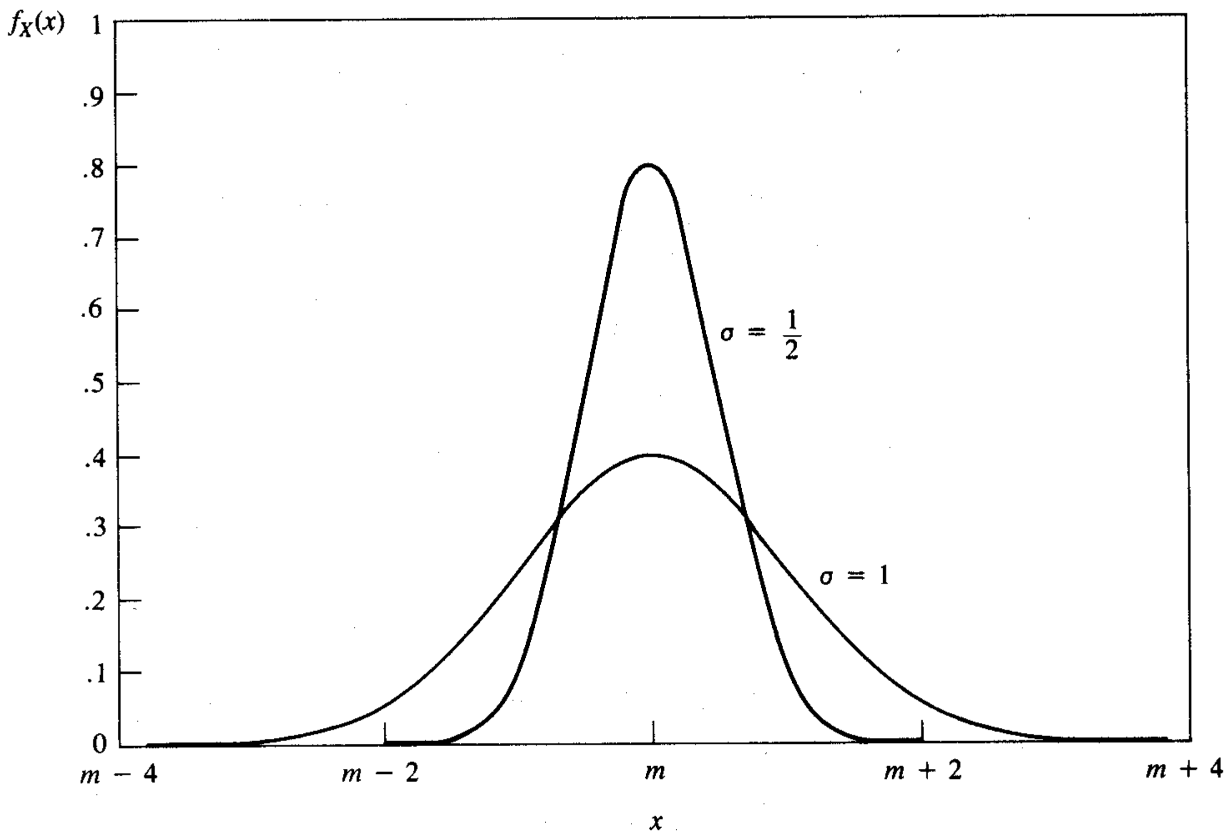
$$P[X \leq x] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-(x'-m)^2/2\sigma^2} dx'.$$

  Change variable $t = (x' - m)/\sigma$ and we have

$$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-m)/\sigma} e^{-t^2/2} dt = \Phi\left(\frac{x-m}{\sigma}\right),$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$



$f_X(x)$
1
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
$\sigma = \frac{1}{2}$
$\sigma = 1$
$m-4$
$m-2$
m
$m+2$
$m+4$
x

Q-function is defined by

$$Q(x) = 1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt.$$

Q-function is the probability of "tail" of the pdf.

$$Q(0) = 1/2 \qquad \text{and} \qquad Q(-x) = 1 - Q(x).$$

$Q(x)$ can be obtained by look-up tables.

Example: A communication system accepts a positive voltage V as input and output a voltage $Y = \alpha V + N$, where $\alpha = 10^{-2}$ and N is a Gaussian random variable with parameters $m = 0$ and $\sigma = 2$. Find the value of V that gives $P[Y < 0] = 10^{-6}$.

Sol:

$$\begin{aligned} P[Y < 0] &= P[\alpha V + N < 0] = P[N < -\alpha V] \\ &= \Phi\left(\frac{-\alpha V}{\sigma}\right) = Q\left(\frac{\alpha V}{\sigma}\right) = 10^{-6}. \end{aligned}$$

From the Q-function table, we have $\alpha V/\sigma = 4.753$. Thus, $V = (4.753)\sigma/\alpha = 950.6$.

3.5 Functions of a Random Variable

- Let X be a random variable. Define another random variable $Y = g(X)$. **Example**: Let the function $h(x) = (x)^+$ be defined as

$$(x)^+ = \begin{cases} 0 & x < 0 \\ x & x \geq 0 \end{cases}.$$

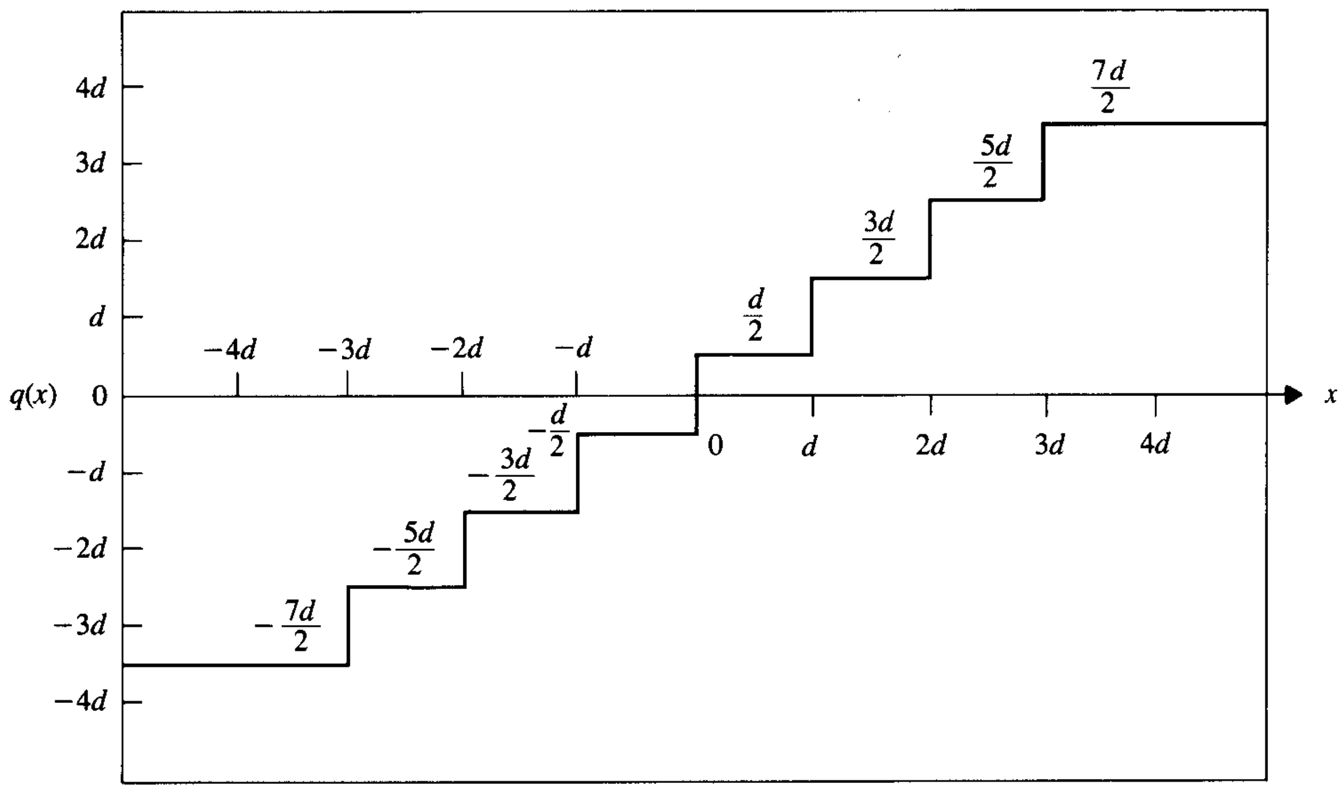
- Let $B = \{x : g(x) \in C\}$. The probability of event C is

$$P[Y \in C] = P[g(X) \in C] = P[X \in B].$$

- Three types of equivalent events are useful in determining the cdf and pdf:

1. Discontinuity case: $\{g(X) = y_k\}$;
2. cdf: $\{g(X) \leq y\}$;
3. pdf: $\{y < g(X) \leq y + h\}$.

Example: Let X be a sample voltage of a speech waveform, and suppose that X has a uniform distribution in the interval $[-4d, 4d]$. Let $Y = q(X)$, where the quantizer input-output characteristic is shown below. Find the pmf for Y.



Sol: The event $\{Y = q\}$ for q in S_Y is equivalent to the event $\{X \in I_q\}$, where I_q is an interval of points mapped into the representation point p. The pmf of Y

$$P[Y = q] = \int_{I_q} f_X(t)dt = 1/8 \qquad \text{for all } q.$$

Example: Let the random variable Y be defined by

$$Y = aX + b,$$

where a is a nonzero constant. Suppose that X has cdf $F_X(x)$, find $F_Y(y)$.

Sol: $\{Y \leq y\}$ and $A = \{aX + b \leq y\}$ are equivalent event. If $a > 0$ then $A = \{X \leq (y-b)/a\}$, and thus

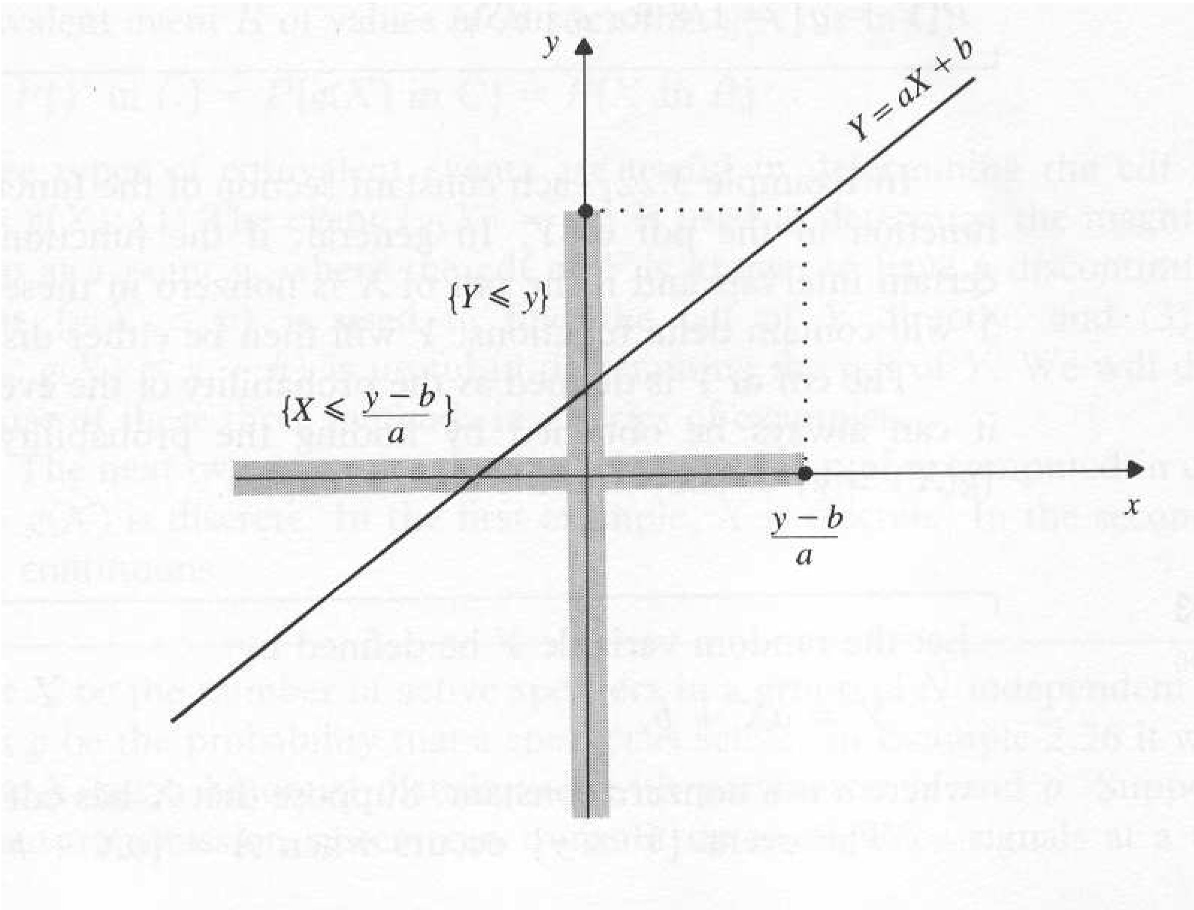
$$F_Y(y) = P\left[X \leq \frac{y-b}{a}\right] = F_X\left(\frac{y-b}{a}\right) \qquad a > 0.$$

If $a < 0$, then $A = \{X \geq (y-b)/a\}$ and

$$F_Y(y) = P\left[X \geq \frac{y-b}{a}\right] = 1 - F_X\left(\frac{y-b}{a}\right).$$

Therefore, we have

$$f_Y(y) = \begin{cases} \frac{1}{a} f_X\left(\frac{y-b}{a}\right) & a > 0 \\ \frac{1}{-a} f_X\left(\frac{y-b}{a}\right) & a < 0 \end{cases} = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$


y
Y = aX + b
{Y ≤ y}
{X ≤ (y − b)/a}
x
(y − b)/a

Example: Let X be a Gaussian random variable with mean m and standard deviation σ:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2} \qquad -\infty < x < \infty$$

Let $Y = aX + b$. Find the pdf of Y.

Sol: From previous example, we have

$$f_Y(y) = \frac{1}{\sqrt{2\pi}|a|\sigma} e^{-(y-b-am)^2/2(a\sigma)^2}.$$

Y also has a Gaussian distribution with mean $am + b$ and standard deviation $|a|\sigma$.

Example: Let random variable Y be defined by

$$Y = X^2,$$

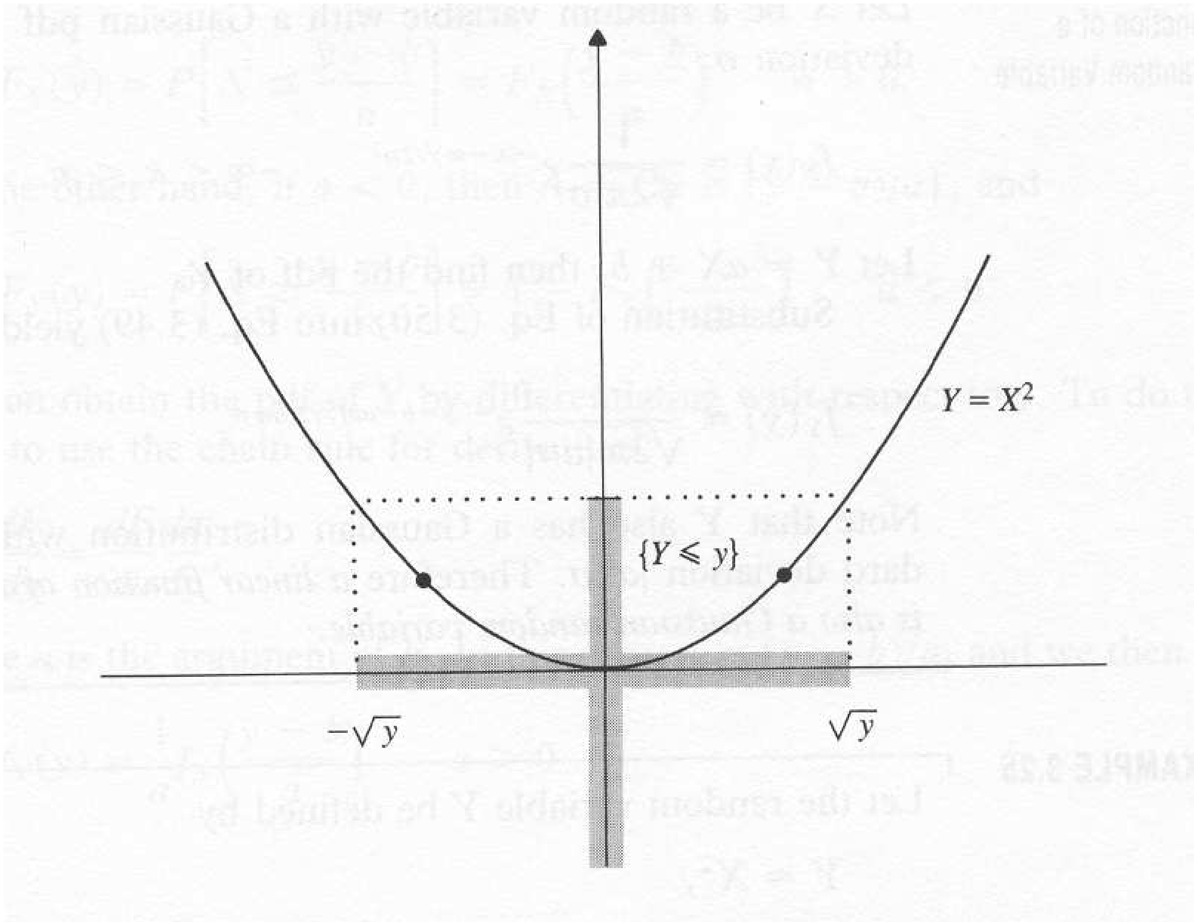
where X is a continuous random variable. Find the cdf and pdf of Y.

Sol: The event $\{Y \leq y\}$ occurs when $\{X^2 \leq y\}$ or equivalently $\{-\sqrt{y} \leq X \leq \sqrt{y}\}$ for y nonnegative. The event is null when y is negative. Then

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ F_X(\sqrt{y}) - F_X(-\sqrt{y}) & y \geq 0 \end{cases}$$

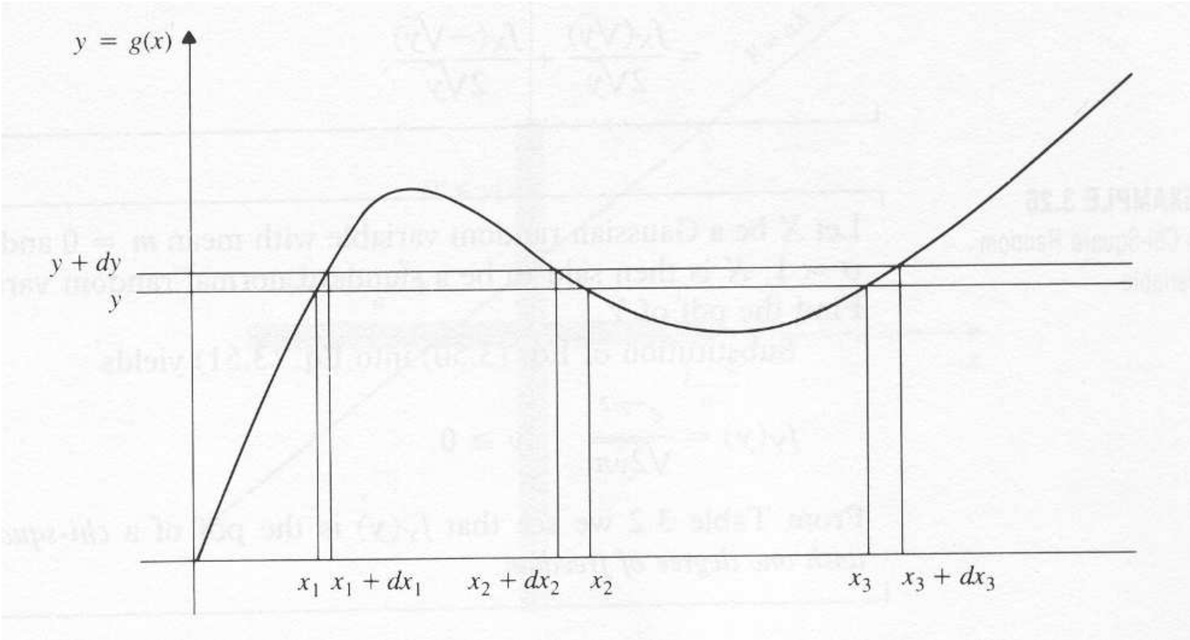
and

$$\begin{aligned} f_Y(y) &= \frac{f_X(\sqrt{y})}{2\sqrt{y}} - \frac{f_X(-\sqrt{y})}{-2\sqrt{y}} \qquad y > 0 \\ &= \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}. \end{aligned}$$



$Y = X^2$
$\{Y \leqslant y\}$
$-\sqrt{y}$
$\sqrt{y}$

- Consider $Y = g(X)$ as shown below



- Consider the event $C_y = \{y < Y < y + dy\}$. Let B_x be its equivalence in the x-axis.
- As shown in the figure, $g(x) = y$ has three solutions and

$$B_x = \{x_1 < X < x_1 + dx_1\} \cup \{x_2 < X < x_2 + dx_2\}$$

$$\cup\{x_3 < X < x_3 + dx_3\}.$$

Thus,

$$P[C_y] = f_Y(y)|dy| = P[B_x] = f_X(x_1)|dx_1| + f_X(x_2)|dx_2| + f_X(x_3)|dx_3|.$$

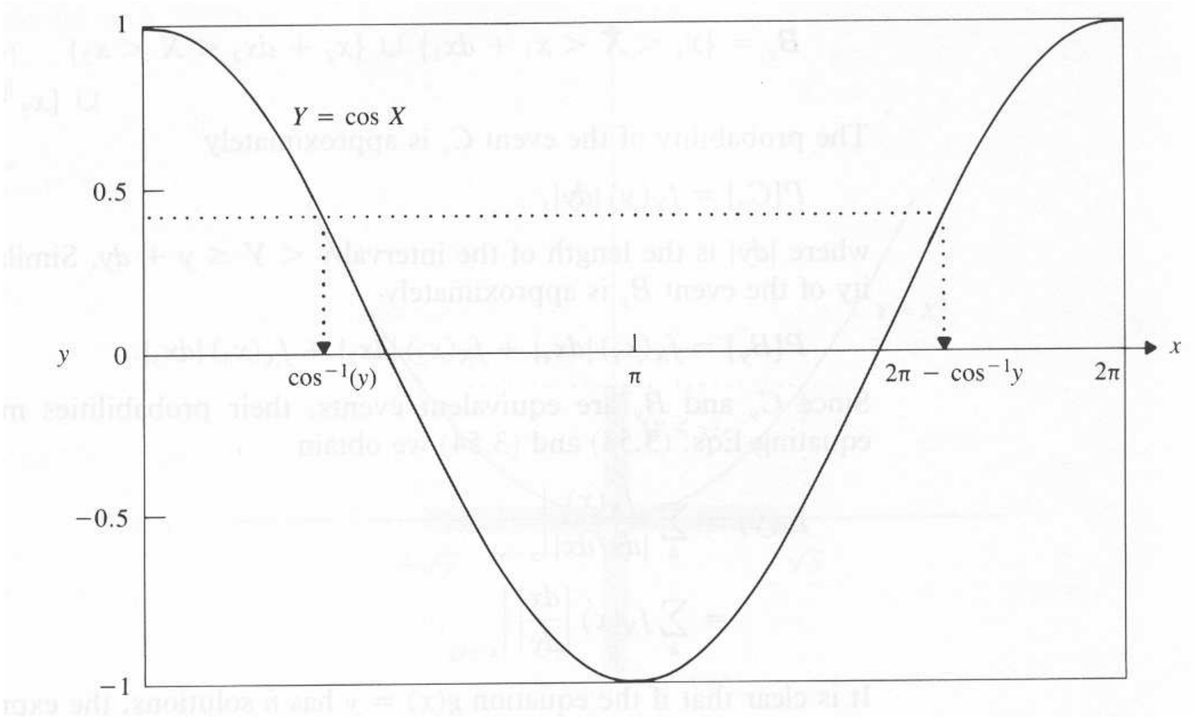
In general, we have

$$f_Y(y) = \sum_k \left. \frac{f_X(x)}{|dy/dx|} \right|_{x=x_k} = \sum_k f_X(x) \left. \left| \frac{dx}{dy} \right| \right|_{x=x_k}.$$

Example : Let $Y = X^2$. For $Y \geq 0$, the equation $y = x^2$ has two solutions, $x_0 = \sqrt{y}$ and $x_1 = -\sqrt{y}$. Since $dy/dx = 2x$, we have

$$f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}.$$

Example: Let $Y = \cos(X)$, where X is uniformly distributed in the interval $(0, 2\pi]$. Find the pdf of Y.



Sol: Two solutions in the interval, $x_0 = \cos^{-1}(y)$ and $x_1 = 2\pi - x_0$.

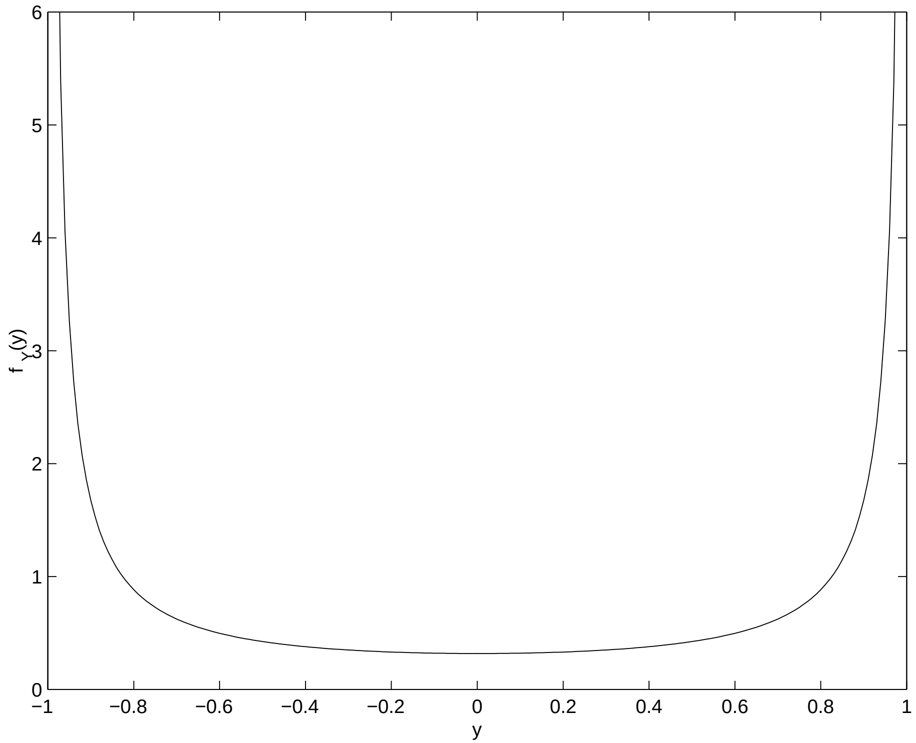
$$\left.\frac{dy}{dx}\right|_{x_0} = -\sin(x_0) = -\sin(\cos^{-1}(y)) = -\sqrt{1-y^2}.$$

Since $f_X(x) = 1/(2\pi)$,

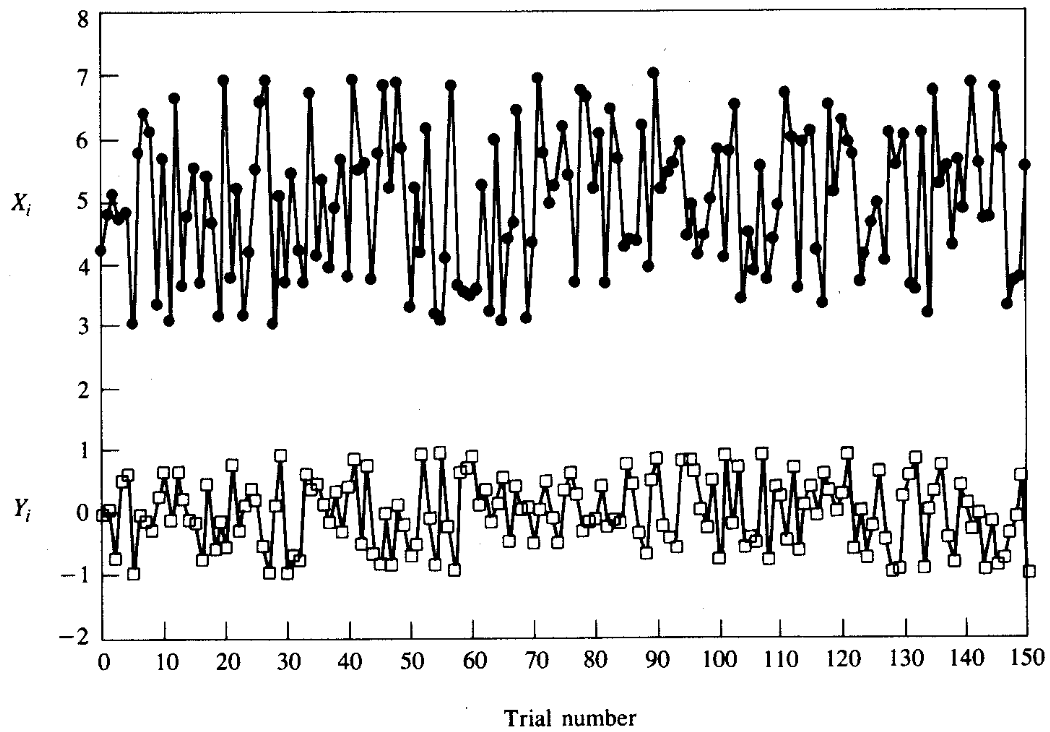
$$\begin{aligned} f_Y(y) &= \frac{1}{2\pi\sqrt{1-y^2}} + \frac{1}{2\pi\sqrt{1-y^2}} \\ &= \frac{1}{\pi\sqrt{1-y^2}} \qquad \text{for } -1 < y < 1. \end{aligned}$$

The cdf of Y is

$$F_Y(y) = \begin{cases} 0 & y < -1 \\ \frac{1}{2} + \frac{\sin^{-1} y}{\pi} & -1 \le y \le 1 \\ 1 & y > 1. \end{cases}$$


6
5
4
3
2
1
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
y
f Y(y)

3.6 Expected Value of Random Variables



The Expected Value of X

- The **expected value** or **mean** of a random variable X is defined by

$$E[X] = \int_{-\infty}^{\infty} t f_X(t) dt$$

- If X is a discrete random variable, then

$$E[X] = \sum_k x_k p_X(x_k)$$

- Note that $E[X]$ may not converge.

- The mean for a uniform random variable between a and b is given by

$$E[X] = \int_a^b \frac{t}{b-a} dt = \frac{a+b}{2}$$

$E[X]$ is the midpoint of the interval $[a, b]$.

- If the pdf of X is symmetric about a point m, then $E[X] = m$. That is, when

$$f_X(m-x) = f_X(m+x),$$

we have

$$0 = \int_{-\infty}^{+\infty} (m-t) f_X(t) dt = m - \int_{-\infty}^{+\infty} t f_X(t) dt.$$

- The pdf of a Gaussian random variable is symmetric at $x = m$. Therefore, $E[X] = m$.

Exercise:

Show that if X is a nonnegative random variable, then

$$E[X] = \int_0^{\infty} (1 - F_X(t))dt \quad \text{if } X \text{ continuous and nonnegative}$$

and

$$E[X] = \sum_{k=0}^{\infty} P[X > k] \quad \text{if } X \text{ nonnegative, integer-valued.}$$

Expected value of $Y = g(X)$

- Let $Y = g(X)$, where X is a random variable with pdf $f_X(x)$.
- Y is also a random variable.
- Mean of Y is

$$E[Y] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx.$$

Variance of X

- Variance of the random variable X is defined by

$$\mathrm{VAR}[X] = E[(X - E[X])^2].$$

- Standard deviation of X

$\mathrm{STD}[X] = \mathrm{VAR}[X]^{1/2}$ −− measure of the spread of a distribution.

- Simplification

$$\begin{aligned}\mathrm{VAR}[X] &= E[X^2 - 2E[X]X + E[X]^2] \\ &= E[X^2] - 2E[X]E[X] + E[X]^2 \\ &= E[X^2] - E[X]^2\end{aligned}$$

Example: Find the variance of the random variable X that is uniformly distributed in the interval $[a, b]$.

$$E[X] = (a + b)/2,$$

and

$$\text{VAR}[X] = \frac{1}{b-a}\int_a^b \left(x - \frac{a+b}{2}\right)^2 dx$$

Let $y = (x - (a + b)/2)$. Then

$$\text{VAR}[X] = \frac{1}{b-a}\int_{-(b-a)/2}^{(b-a)/2} y^2 dy = \frac{(b-a)^2}{12}.$$

Example: Find the variance of a Gaussian random variable.

Multiply the integral of the pdf of X by $\sqrt{2\pi}\sigma$ to obtain

$$\int_{-\infty}^{+\infty} e^{-(x-m)^2/2\sigma^2} dx = \sqrt{2\pi}\sigma.$$

Differentiate both sides with respect to σ to get

$$\int_{-\infty}^{+\infty} \left(\frac{(x-m)^2}{\sigma^3}\right) e^{-(x-m)^2/2\sigma^2} dx = \sqrt{2\pi}.$$

Then

$$\text{VAR}[X] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} (x-m)^2 e^{-(x-m)^2/2\sigma^2} dx = \sigma^2.$$

- **Properties**

  Let c be a constant. Then

$$\begin{aligned}\text{VAR}[c] &= 0,\\ \text{VAR}[X+c] &= \text{VAR}[X],\\ \text{VAR}[cX] &= c^2\text{VAR}[X].\end{aligned}$$

- nth moment of the random variable X is given by

$$E[X^n] = \int_{-\infty}^{+\infty} x^n f_X(x)dx.$$

3.7 Markov and Chebyshev Inequalities

Markov Inequality

- Suppose X is a nonnegative random variable with mean $E[X]$. Then

$$P[X \geq a] \leq \frac{E[X]}{a} \qquad \text{for } X \text{ nonnegative}$$

Since

$$\begin{aligned} E[X] &= \int_0^a t f_X(t)dt + \int_a^\infty t f_X(t)dt \geq \int_a^\infty t f_X(t)dt \\ &\geq \int_a^\infty a f_X(t)dt = aP[X \geq a]. \end{aligned}$$

Chebyshev Inequality

- Consider random variable X with $E[X] = m$ and $\text{VAR}[X] = \sigma^2$. Then
$$P[|X - m| \geq a] \leq \frac{\sigma^2}{a^2}.$$
- Proof: Let $D^2 = (X - m)^2$. Markov inequality for D^2 gives
$$P[D^2 \geq a^2] \leq \frac{E[(X - m)^2]}{a^2} = \frac{\sigma^2}{a^2}.$$
- $\{D^2 \geq a^2\}$ and $\{|X - m| \geq a\}$ are equivalent events.

3.9 Transfer Methods

The Characteristic Function

- The characteristic function of a random variable X is defined by

$$\begin{aligned}\mathbf{\Phi}_X(\omega) &= E\left[e^{j\omega X}\right] \\ &= \int_{-\infty}^{\infty} f_X(x)e^{j\omega x}\, dx,\end{aligned}$$

  where $j = \sqrt{-1}$ is the imaginary unit number.

- $\mathbf{\Phi}_X(\omega)$ can be viewed as the expected value of a function of X, $e^{j\omega X}$.

- $\mathbf{\Phi}_X(\omega)$ is the Fourier transform of the pdf $f_X(x)$ with a reversal in the sign of the exponent.
- From the Fourier transform inversion formula we have

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{\Phi}_X(\omega) e^{-j\omega x} \, d\omega.$$

Example: The characteristic function for an exponentially distributed random variable with parameter λ is given by

$$\begin{aligned}\mathbf{\Phi}_X(\omega) &= \int_0^\infty \lambda e^{-\lambda x} e^{j\omega x}\ dx = \int_0^\infty \lambda e^{-(\lambda - j\omega)x}\ dx \\ &= \frac{\lambda}{\lambda - j\omega}.\end{aligned}$$

- If X is a discrete random variable, we have

$$\mathbf{\Phi}_X(\omega) = \sum_k p_X(x_k) e^{j\omega x_k}.$$

- If X is an integer-valued random variable, we have

$$\mathbf{\Phi}_X(\omega) = \sum_{k=-\infty}^{\infty} p_X(k) e^{j\omega k}.$$

- The above is the Fourier transform of the sequence $p_X(k)$.
- It is a periodic function of ω with period 2π.
- By the inversion formula we have

$$p_X(k) = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{\Phi}_X(\omega) e^{-j\omega k}\, d\omega \quad k = 0,\ \pm 1,\ \pm 2, \ldots.$$

Example: The characteristic function for a geometric random variable is given by

$$\begin{aligned}\mathbf{\Phi}_X(\omega) &= \sum_{k=0}^{\infty} pq^k e^{j\omega k} = p\sum_{k=0}^{\infty}\left(qe^{j\omega}\right)^k \\ &= \frac{p}{1-qe^{j\omega}}.\end{aligned}$$

- The **moment theorem** states that the moments of X are given by

$$E\left[X^n\right] = \frac{1}{j^n}\frac{d^n}{d\omega^n}\mathbf{\Phi}_X(\omega)\Big|_{\omega=0}.$$

Proof: First we expend $e^{j\omega x}$ in power series in the definition of $\mathbf{\Phi}_X(\omega)$:

$$\mathbf{\Phi}_X(\omega) = \int_{-\infty}^{\infty} f_X(x)\left\{1 + j\omega X + \frac{(j\omega X)^2}{2!} + \cdots\right\} dx.$$

Assuming that all the moments of X are finite and that the series can be integrated term by term, we have

$$\begin{aligned}\mathbf{\Phi}_X(\omega) &= 1 + j\omega E[X] + \frac{(j\omega)^2 E\left[X^2\right]}{2!} + \cdots \\ &\quad + \frac{(j\omega)^n E\left[X^n\right]}{n!} + \cdots.\end{aligned}$$

If we differentiate n times and evaluate at $\omega = 0$, we have

$$\frac{d^n}{d\omega^n}\mathbf{\Phi}_X(\omega)\bigg|_{\omega=0} = j^n E\left[X^n\right].$$

Example: To find the mean of an exponentially distributed random variable, we differentiate $\mathbf{\Phi}_X(\omega) = \lambda(\lambda - j\omega)^{-1}$ once, and obtain

$$\mathbf{\Phi}'_X(\omega) = \frac{\lambda j}{(\lambda - j\omega)^2}.$$

Then $E[X] = \mathbf{\Phi}'_X(0)/j = 1/\lambda$.

The Probability Generating Function

- The probability generating function $G_N(z)$ of a nonnegative integer-valued random variable N is defined by

$$G_N(z) = E\left[z^N\right] = \sum_{k=0}^{\infty} p_N(k) z^k.$$

- $G_N(z)$ can be viewed as the expected value of a function of N, z^N.
- $G_N(z)$ is the z-transform of the pmf $p_N(k)$ with a sign change in the exponent.
- Similar to the derivation of the moment theorem, we have

$$p_N(k) = \frac{1}{k!}\frac{d^k}{dz^k} G_N(z)\Big|_{z=0}.$$

- $$\frac{d}{dz}G_N(z)\Big|_{z=1} = \sum_{k=0}^{\infty} p_N(k)kz^{k-1}\Big|_{z=1} = \sum_{k=0}^{\infty} kp_N(k) = E[N].$$

- $$\begin{aligned}\frac{d^2}{dz^2}G_N(z)\Big|_{z=1} &= \sum_{k=0}^{\infty} p_N(k)k(k-1)z^{k-2}\Big|_{z=1} \\ &= \sum_{k=0}^{\infty} k(k-1)p_N(k) = E[N(N-1)] \\ &= E\left[N^2\right] - E[N].\end{aligned}$$

- Thus, the mean and variance of N are given by

$$E[N] = G'_N(1)$$

and

$$VAR[N] = G''_N(1) + G'_N(1) - (G'_N(1))^2.$$

Example: The probability generating function for the Poisson random variable with parameter α is given by

$$\begin{aligned} G_N(z) &= \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} e^{-\alpha} z^k = e^{-\alpha} \sum_{k=0}^{\infty} \frac{(\alpha z)^k}{k!} \\ &= e^{-\alpha} e^{\alpha z} = e^{\alpha(z-1)}. \end{aligned}$$

The first two derivatives of $G_N(z)$ are given by

$$G'_N(z) = \alpha e^{\alpha(z-1)}$$

and

$$G''_N(z) = \alpha^2 e^{\alpha(z-1)}.$$

Therefore,

$$E[N] = \alpha \text{ and } VAR[N] = \alpha^2 + \alpha - \alpha^2 = \alpha.$$

The Laplace Transform of the pdf

- The Laplace transform of the pdf is given by

$$X^*(s) = \int_0^\infty f_X(x)e^{-sx}\, dx = E\left[e^{-sX}\right].$$

- $X^*(s)$ can be viewed as an expected value of a function of X, e^{-sX}.
- The moment theorem also holds for $X^*(s)$:

$$E[X^n] = (-1)^n \frac{d^n}{ds^n} X^*(s)\Big|_{s=0}.$$