Chapter 6: Random Processes[1]

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[1] Modified from the lecture notes by Prof. Mao-Ching Chiu

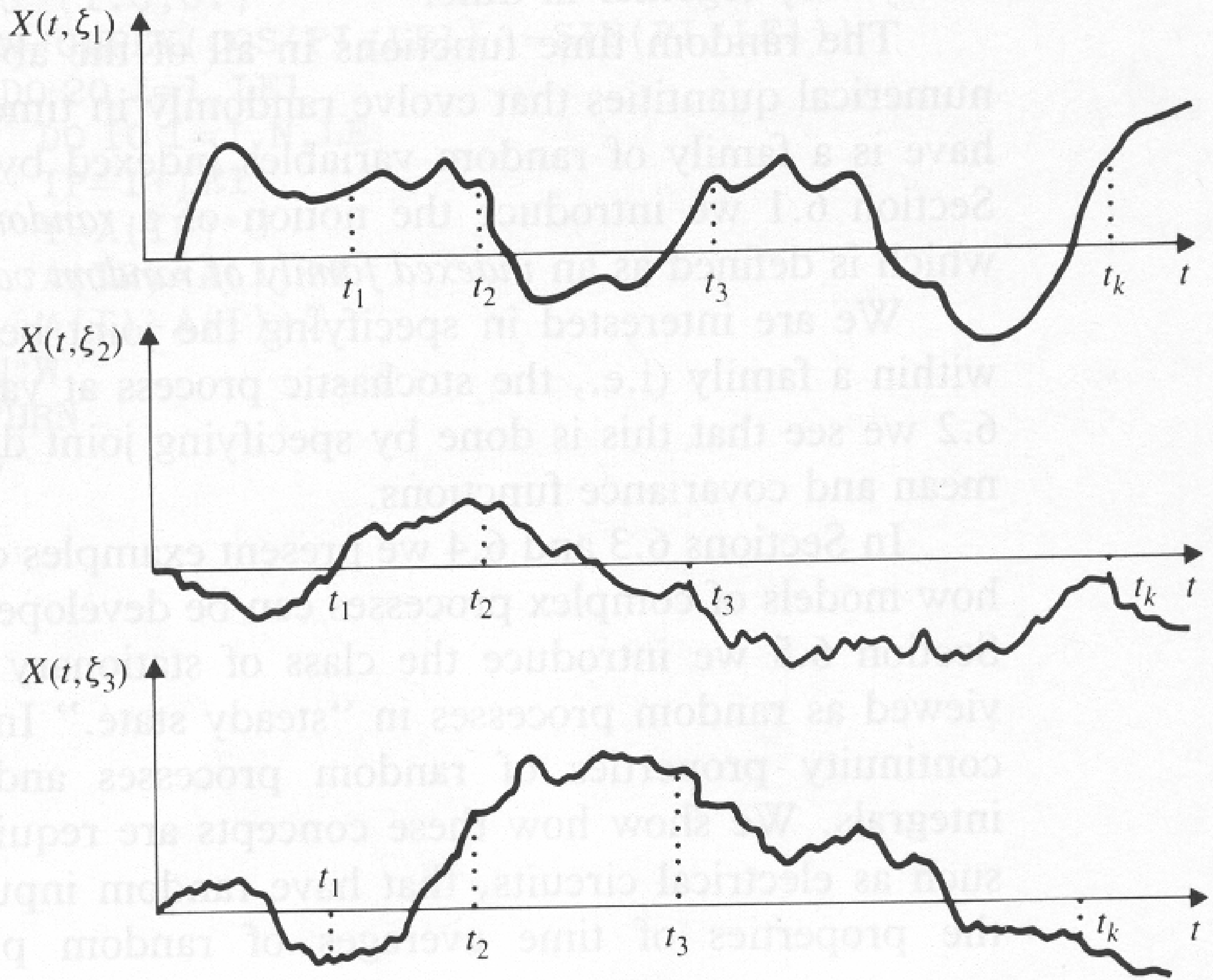
Definition of a Random Process

- Random experiment with sample space S.
- To every outcome $\zeta \in S$, we assign a function of time according to some rule:

$$X(t, \zeta) \qquad t \in I.$$

- For fixed ζ, the graph of the function $X(t, \zeta)$ versus t is a sample function of the random process.
- For each fixed t_k from the index set I, $X(t_k, \zeta)$ is a random variable.

- The indexed family of random variables $\{X(t, \zeta), t \in I\}$ is called a **random process** or **stochastic process**.



$X(t,\xi_1)$
t_1
t_2
t_3
t_k
t
$X(t,\xi_2)$
t_1
t_2
t_3
t_k
t
$X(t,\xi_3)$
t_1
t_2
t_3
t_k
t

- A stochastic process is said to be **discrete-time** if the index set I is a countable set.
- A **continuous-time** stochastic process is one in which I is continuous.

Example: Let ζ be a number selected at random from the interval $S = [0, 1]$, and let $b_1 b_2 \cdots$ be the binary expansion of ζ

$$\zeta = \sum_{i=1}^{\infty} b_i 2^{-i} \qquad b_i \in \{0, 1\}.$$

Define the discrete-time random process $X(n, \zeta)$ by

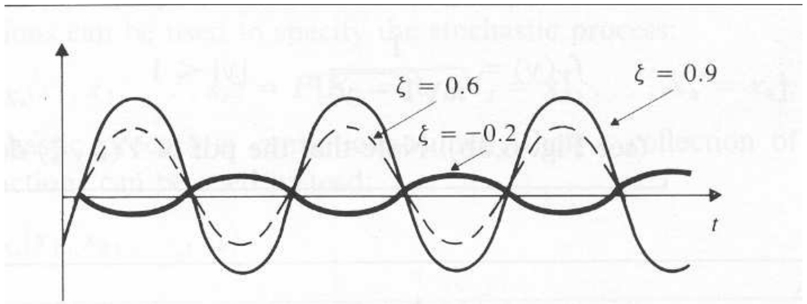
$$X(n, \zeta) = b_n \qquad n = 1, 2, \cdots.$$

A sequence of binary numbers is obtained.

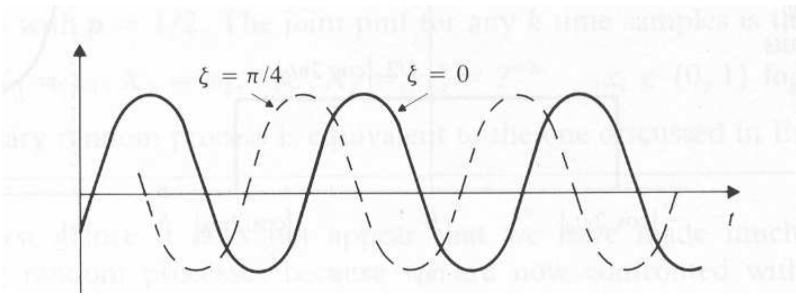
Example:

1. Let $\zeta \in S = [-1, +1]$ be selected at random. Define the continuous-time random process $X(t, \zeta)$ by

$$X(t, \zeta) = \zeta \cos(2\pi t) \qquad -\infty < t < \infty.$$



2. Let $\zeta \in S = (-\pi, \pi)$ be selected at random, and let $Y(t, \zeta) = cos(2\pi t + \zeta)$


ζ = π/4
ζ = 0
t

6.2 Specifying a Random Process

Joint Distributions of Time Samples

- Let $X_1, X_2, \ldots, X_k$ be the k random variables obtained by sampling the random process $X(t, \zeta)$ at the time $t_1, t_2, \ldots, t_k$:

$$X_1 = X(t_1, \zeta), \quad X_2 = X(t_2, \zeta), \ldots, \quad X_k = X(t_k, \zeta).$$

- The joint behavior of the random process at these k time instants is specified by the joint cdf of $(X_1, X_2, \ldots, X_k)$.

A stochastic process is specified by the collection of kth-order joint cumulative distribution functions:

$$F_{X_1,\ldots,X_k}(x_1,\ldots,x_k) = P[X_1 \le x_1,\ldots,X_k \le x_k]$$

for any k and any choice of sampling instants $t_1,\ldots,t_k$.

- If the stochastic process is discrete-valued, then a collection of probability mass functions can be used to specify the stochastic process

$$p_{X_1,\ldots,X_k}(x_1,\ldots,x_k) = P[X_1 = x_1,\ldots,X_k = x_k].$$

- If the stochastic process is continuous-valued, then a collection of probability density functions can be used instead:

$$f_{X_1,\ldots,X_k}(x_1,\ldots,x_k).$$

Example: Let X_n be a sequence of independent, identically distributed Bernoulli random variables with $p = 1/2$. The joint pmf for any k time samples is then

$$P[X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k] = 2^{-k} \quad x_i \in \{0, 1\} \;\; \forall\, i.$$

- A random process $X(t)$ is said to have **independent increments** if for any k and any choice of sampling instants $t_1 < t_2 \cdots < t_k$, the random variables

$$X(t_2) - X(t_1), X(t_3) - X(t_2), \ldots, X(t_k) - X(t_{k-1})$$

are independent random variables.

- A random process $X(t)$ is said to be **Markov** if the future of the process given the present is independent of the past; that is, for any k and any choice of sampling instants $t_1 < t_2 < \cdots < t_k$ and for any $x_1, x_2, \ldots, x_k$,

$$\begin{aligned} & f_{X(t_k)}(x_k | X(t_{k-1}) = x_{k-1}, \ldots, X(t_1) = x_1) \\ &= f_{X(t_k)}(x_k | X(t_{k-1}) = x_{k-1}) \end{aligned}$$

if $X(t)$ is continuous-valued, and

$$P[X(t_k) = x_k | X(t_{k-1}) = x_{k-1}, \ldots, X(t_1) = x_1)$$
$$= P[X(t_k) = x_k | X(t_{k-1}) = x_{k-1})$$

if $X(t)$ is discrete-valued.

Independent increments $\rightarrow$ Markov;

Markov $\overrightarrow{\text{NOT}}$ independent increments

The Mean, Autocorrelation, and Autocovariance Functions

- The **mean** $m_X(t)$ of a random process $X(t)$ is defined by
$$m_X(t) = E[X(t)] = \int_{-\infty}^{+\infty} x f_{X(t)}(x)dx,$$
where $f_{X(t)}(x)$ is the pdf of $X(t)$.
- $m_X(t)$ is a function of time.
- The **autocorrelation** $R_X(t_1, t_2)$ of a random process

$X(t)$ is defined as

$$\begin{aligned} R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xyf_{X(t_1),X(t_2)}(x, y)dxdy. \end{aligned}$$

- In general, the autocorrelation is a function of t_1 and t_2.

- The autocovariance $C_X(t_1, t_2)$ of a random process $X(t)$ is defined as the covariance of $X(t_1)$ and $X(t_2)$

$$C_X(t_1, t_2) = E\left[\{X(t_1) - m_X(t_1)\}\{X(t_2) - m_X(t_2)\}\right].$$

- $$C_X(t_1, t_2) = R_X(t_1, t_2) - m_X(t_1)m_X(t_2).$$

- The variance of $X(t)$ can be obtained from $C_X(t_1, t_2)$:

$$\text{VAR}[X(t)] = E[(X(t) - m_X(t))^2] = C_X(t, t).$$

- The correlation coefficient of $X(t)$ is given by

$$\rho_X(t_1, t_2) = \frac{C_X(t_1, t_2)}{\sqrt{C_X(t_1, t_1)}\sqrt{C_X(t_2, t_2)}}.$$

- $|\rho_X(t_1, t_2)| \leq 1$.

Example: Let $X(t) = A\cos 2\pi t$, where A is some random variable. The mean of $X(t)$ is given by

$$m_X(t) = E[A\cos 2\pi t] = E[A]\cos 2\pi t.$$

The autocorrelation is

$$\begin{aligned} R_X(t_1, t_2) &= E[A\cos(2\pi t_1)A\cos(2\pi t_2)] \\ &= E[A^2]\cos(2\pi t_1)\cos(2\pi t_2), \end{aligned}$$

and the autocovariance

$$\begin{aligned} C_X(t_1, t_2) &= R_X(t_1, t_2) - m_X(t_1)m_X(t_2) \\ &= \{E[A^2] - E[A]^2\}\cos(2\pi t_1)\cos(2\pi t_2) \\ &= \mathrm{VAR}[A]\cos(2\pi t_1)\cos(2\pi t_2). \end{aligned}$$

Example: Let $X(t) = \cos(\omega t + \Theta)$, where Θ is uniformly distributed in the interval $(-\pi, \pi)$. The mean of $X(t)$ is given by

$$m_X(t) = E[\cos(\omega t + \Theta)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\omega t + \theta) d\theta = 0.$$

The autocorrelation and autocovariance are then

$$\begin{aligned} C_X(t_1, t_2) &= R_X(t_1, t_2) = E[\cos(\omega t_1 + \Theta) \cos(\omega t_2 + \Theta)] \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} \{\cos(\omega(t_1 - t_2)) + \cos(\omega(t_1 + t_2) + 2\theta)\} d\theta \\ &= \frac{1}{2} \cos(\omega(t_1 - t_2)). \end{aligned}$$

Gaussian Random Process A random process $X(t)$ is a **Gaussian random process** if the samples $X_1 = X(t_1), X_2 = X(t_2), \ldots, X_k = X(t_k)$ are joint Gaussian random variables for all k, and all choices of $t_1, \ldots, t_k$:

$$f_{X_1,X_2,\ldots,X_k}(x_1,\ldots,x_k) = \frac{e^{-1/2(\boldsymbol{x}-\boldsymbol{m})K^{-1}(\boldsymbol{x}-\boldsymbol{m})}}{(2\pi)^{k/2}|K|^{1/2}},$$

where

$$\boldsymbol{m} = \begin{bmatrix} m_X(t_1) \\ \vdots \\ m_X(t_k) \end{bmatrix} \quad K = \begin{bmatrix} C_X(t_1,t_1) & C_X(t_1,t_2) & \cdots & C_X(t_1,t_k) \\ C_X(t_2,t_1) & C_X(t_2,t_2) & \cdots & C_X(t_2,t_k) \\ \vdots & \vdots & & \vdots \\ C_X(t_k,t_1) & \cdots & & C_X(t_k,t_k) \end{bmatrix}.$$

The joint pdf's of Gaussian random process are completely specified by the mean and by covariance function.

Linear operation on a Gaussian random process results in another Gaussian random process.

Example: Let the discrete-time random process X_n be a sequence of independent Gaussian random variables with mean m and variance σ^2. The covariance matrix for the time $t_1, \ldots, t_k$ is

$$\{C_X(t_i, t_j)\} = \{\sigma^2 \delta_{ij}\} = \sigma^2 I,$$

where $\delta_{ij} = 1$ when $i = j$ and 0 otherwise, and I is the identity matrix.

The corresponding joint pdf

$$\begin{aligned} f_{X_1,\ldots,X_k}(x_1, \ldots, x_k) &= \frac{1}{(2\pi\sigma^2)^{k/2}} \exp\left\{ -\sum_{i=1}^{k} (x_i - m)^2 / 2\sigma^2 \right\} \\ &= f_X(x_1) f_X(x_2) \cdots f_X(x_k). \end{aligned}$$

Multiple Random Processes

- The joint behavior of $X(t)$ and $Y(t)$ must specify all possible joint density functions of $X(t_1), \ldots, X(t_k)$ and $Y(t'_1), \ldots, Y(t'_j)$ for all k, j and all choices of $t_1, \ldots, t_k$ and $t'_1, \ldots, t'_j$.
- $X(t)$ and $Y(t)$ are said to be **independent** if the vector random variables $(X(t_1), \ldots, X(t_k))$ and $(Y(t'_1), \ldots, Y(t'_j))$ are independent for all k, j and all choices of $t_1, \ldots, t_k$ and $t'_1, \ldots, t'_j$.
- The **cross-correlation** $R_{X,Y}(t_1, t_2)$ of $X(t)$ and $Y(t)$ is defined by
$$R_{X,Y}(t_1, t_2) = E[X(t_1)Y(t_2)].$$
- The process $X(t)$ and $Y(t)$ are said to be **orthogonal** if
$$R_{X,Y}(t_1, t_2) = 0 \quad \text{for all } t_1 \text{ and } t_2.$$

- The **cross-covariance** $C_{X,Y}(t_1, t_2)$ of $X(t)$ and $Y(t)$ is defined by

$$\begin{aligned} C_{X,Y}(t_1, t_2) &= E[\{X(t_1) - m_X(t_1)\}\{Y(t_2) - m_Y(t_2)\}] \\ &= R_{X,Y}(t_1, t_2) - m_X(t_1)m_Y(t_2). \end{aligned}$$

- The process $X(t)$ and $Y(t)$ are said to be **uncorrelated** if

$$C_{X,Y}(t_1, t_2) = 0 \quad \text{for all } t_1 \text{ and } t_2.$$

- Note that

$$\begin{aligned} & C_{X,Y}(t_1, t_2) = 0 \\ \leftrightarrow \quad & R_{X,Y}(t_1, t_2) = E[X(t_1)Y(t_2)] = m_X(t_1)m_Y(t_2) = E[X(t_1)]E[Y(t_2)]. \end{aligned}$$

Example: Let $X(t) = \cos(\omega t + \Theta)$ and $Y(t) = \sin(\omega t + \Theta)$, where Θ is a random variable uniformly distributed in $[-\pi, \pi]$. Find the cross-covariance of $X(t)$ and $Y(t)$.

Sol: Since $X(t)$ and $Y(t)$ are zero-mean, the cross-covariance is equal to the cross-correlation.

$$
\begin{aligned}
R_{X,Y}(t_1, t_2) &= E[\cos(\omega t_1 + \Theta)\sin(\omega t_2 + \Theta)] \\
&= E\left[-\frac{1}{2}\sin(\omega(t_1 - t_2)) + \frac{1}{2}\sin(\omega(t_1 + t_2) + 2\Theta)\right] \\
&= -\frac{1}{2}\sin(\omega(t_1 - t_2)),
\end{aligned}
$$

since $E[\sin(\omega(t_1 + t_2) + 2\Theta)] = 0$.

Example: Suppose we observe a process $Y(t)$:

$$Y(t) = X(t) + N(t).$$

Find the cross-correlation between the observed signal and the desired signal assuming that $X(t)$ and $N(t)$ are independent random processes.

$$\begin{aligned} R_{X,Y}(t_1, t_2) &= E[X(t_1)Y(t_2)] \\ &= E[X(t_1)\{X(t_2) + N(t_2)\}] \\ &= E[X(t_1)X(t_2)] + E[X(t_1)N(t_2)] \\ &= R_X(t_1, t_2) + E[X(t_1)]E[N(t_2)] \\ &= R_X(t_1, t_2) + m_X(t_1)m_N(t_2). \end{aligned}$$

6.3 Examples of Discrete-Time Random Processes

iid Random Processes

- The sequence X_n is called independent, identically distributed **(iid) random process**, if the joint cdf for any time instants $n_1, \ldots, n_k$ can be expressed as

$$\begin{aligned} F_{X_{n_1},\ldots,X_{n_k}}(x_{n_1}, \ldots, x_{n_k}) &= P[X_{n_1} \le x_{n_1}, \ldots, X_{n_k} \le x_{n_k}] \\ &= F_{X_{n_1}}(x_{n_1}) \cdots F_{X_{n_k}}(x_{n_k}). \end{aligned}$$

- The mean of an iid process is

$$m_X(n) = E[X_n] = m \quad \text{for all } n.$$

- Autocovariance function:
  - If $n_1 \neq n_2$

$$\begin{aligned} C_X(n_1, n_2) &= E[(X_{n_1} - m)(X_{n_2} - m)] \\ &= E[(X_{n_1} - m)]E[(X_{n_2} - m)] = 0 \end{aligned}$$

    since X_{n_2} and X_{n_2} are independent.
  - If $n_1 = n_2 = n$

$$C_X(n_1, n_2) = E[(X_n - m)^2] = \sigma^2.$$

  - Therefore

$$C_X(n_1, n_2) = \sigma^2 \delta_{n_1, n_2}.$$

- The autocorrelation function of an iid process

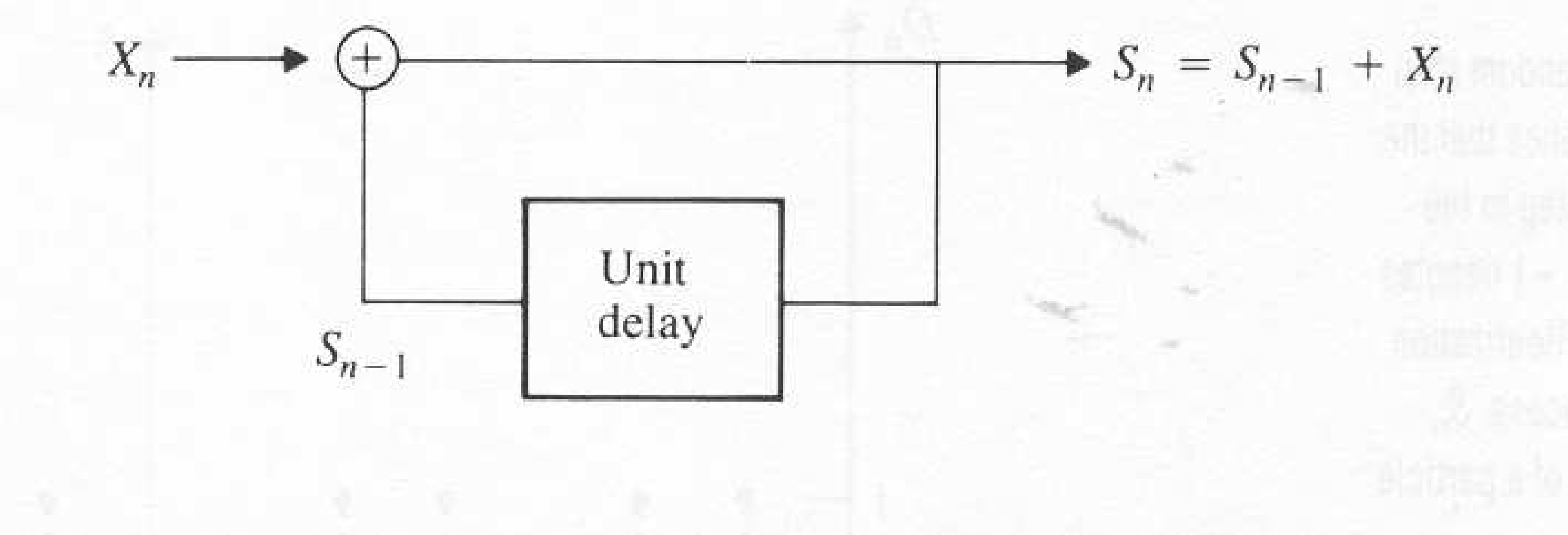
$$R_X(n_1, n_2) = C_X(n_1, n_2) + m^2.$$

Sum Processes: The Binomial Counting and Random Walk Process

- Consider a random process S_n which is the sum of a sequence of iid random variables, $X_1, X_2, \ldots$:

$$\begin{aligned} S_n &= X_1 + X_2 + \ldots + X_n \quad n = 1, 2, \ldots \\ &= S_{n-1} + X_n, \end{aligned}$$

where $S_0 = 0$.



- We call S_n the sum process. S_n is independent of the past when S_{n-1} is known.

Example: Let I_i be the sequence of independent Bernoulli random variable, and let S_n be the corresponding sum process. S_n is a binomial random variable with parameter n and $p = P[I = 1]$:

$$P[S_n = j] = \binom{n}{j} p^j (1-p)^{n-j} \quad \text{for } 0 \leq j \leq n,$$

and zero otherwise. Thus S_n has mean np and variance $np(1-p)$.

- The sum process S_n has **independent increments** in nonoverlapping time intervals.
- For example: $n_0 < n \leq n_1$ and $n_2 < n \leq n_3$, where $n_1 \leq n_2$. We have

$$\begin{aligned} S_{n_1} - S_{n_0} &= X_{n_0+1} + \cdots + X_{n_1} \\ S_{n_3} - S_{n_2} &= X_{n_2+1} + \cdots + X_{n_3}. \end{aligned}$$

  The independence of the X_n's implies $(S_{n_1} - S_{n_0})$ and $(S_{n_3} - S_{n_2})$ are independent random variables.
- For $n' > n$, $(S_{n'} - S_n)$ is the sum of $n' - n$ iid random variables, so it has the same distribution as $S_{n'-n}$

$$P[S_{n'} - S_n = y] = P[S_{n'-n} = y].$$

- Thus, increments in intervals of the same length have the same distribution regardless of when the interval begins. We say that S_n has **stationary increments**.

- Compute the joint pmf of S_n at time n_1, n_2, and n_3

$$\begin{aligned} & P[S_{n_1} = y_1, S_{n_2} = y_2, S_{n_3} = y_3] \\ = \; & P[S_{n_1} = y_1, S_{n_2} - S_{n_1} = y_2 - y_1, S_{n_3} - S_{n_2} = y_3 - y_2] \\ = \; & P[S_{n_1} = y_1]P[S_{n_2} - S_{n_1} = y_2 - y_1] \\ & \times P[S_{n_3} - S_{n_2} = y_3 - y_2]. \end{aligned}$$

- The stationary increments property implies that

$$\begin{aligned} & P[S_{n_1} = y_1, S_{n_2} = y_2, S_{n_3} = y_3] \\ & = P[S_{n_1} = y_1]P[S_{n_2 - n_1} = y_2 - y_1]P[S_{n_3 - n_2} = y_3 - y_2]. \end{aligned}$$

- In general, we have

$$P[S_{n_1} = y_1, S_{n_2} = y_2, \ldots, S_{n_k} = y_k]$$

$$= P[S_{n_1} = y_1]P[S_{n_2-n_1} = y_2 - y_1] \cdots P[S_{n_k-n_{k-1}} = y_k - y_{k-1}].$$

- If X_n are continuous-valued random variables, then

$$f_{S_{n_1},\ldots,S_{n_k}}(y_1, \ldots, y_k) = f_{S_{n_1}}(y_1)f_{S_{n_2-n_1}}(y_2 - y_1) \cdots f_{S_{n_k}-S_{n_{k-1}}}(y_k - y_{k-1}).$$

Example: Find the joint pmf for the binomial counting process at times n_1 and n_2.

$$P[S_{n_1} = y_1, S_{n_2} = y_2] = P[S_{n_1} = y_1]P[S_{n_2-n_1} = y_2 - y_1] = \binom{n_2 - n_1}{y_2 - y_1} p^{y_2-y_1}(1-p)^{n_2-n_1-y_2+y_1}$$

$$\times \binom{n_1}{y_1} p^{y_1}(1-p)^{n_1-y_1}$$

$$= \binom{n_2-n_1}{y_2-y_1} \binom{n_1}{y_1} p^{y_2}(1-p)^{n_2-y_2}.$$

- The mean and variance of a sum process

$$m_S(n) = E[S_n] = nE[X] = nm$$
$$\text{VAR}[S_n] = n\text{VAR}[X] = n\sigma^2.$$

- The autocovariance of S_n

$$\begin{aligned} C_S(n,k) &= E[(S_n - E[S_n])(S_k - E[S_k])] \\ &= E[(S_n - nm)(S_k - km)] \\ &= E\left[\left\{\sum_{i=1}^{n}(X_i - m)\right\}\left\{\sum_{j=1}^{k}(X_j - m)\right\}\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{k} E[(X_i - m)(X_j - m)] \end{aligned}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{k} C_X(i,j)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{k} \sigma^2 \delta_{i,j}$$

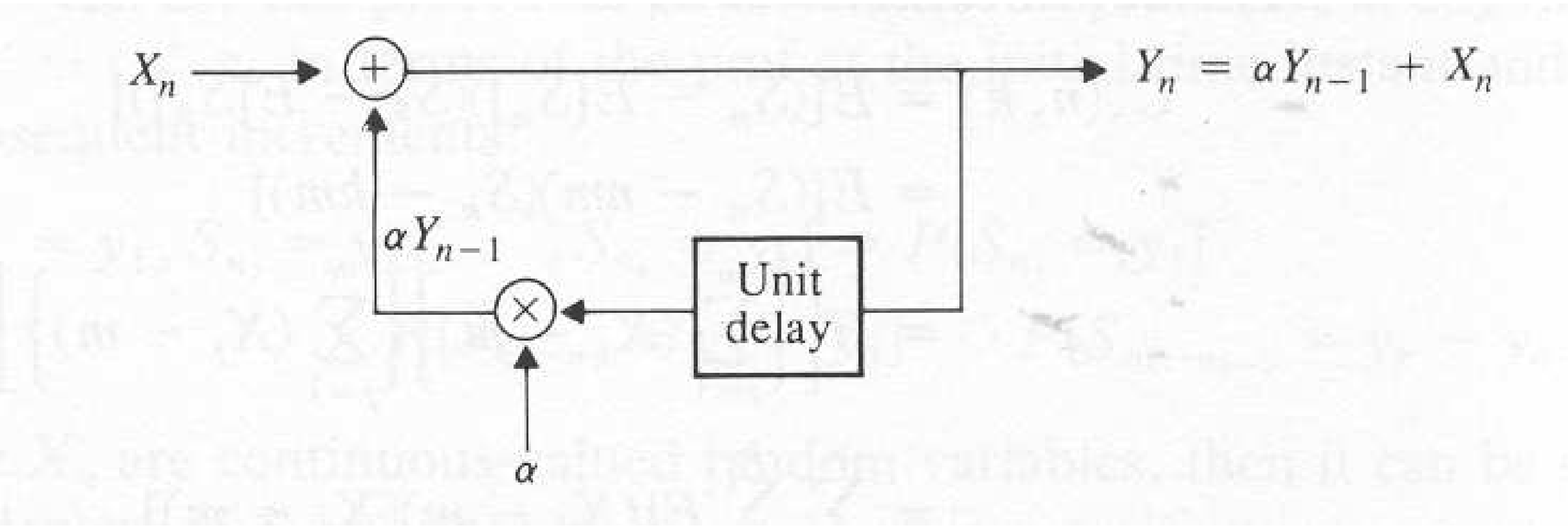
$$= \sum_{i=1}^{\min(n,k)} C_X(i,i) = \min(n,k)\sigma^2.$$

- The property of independent increments allows us to compute the autocovariance in another way.
- Suppose $n \leq k$ so $n = \min(n,k)$

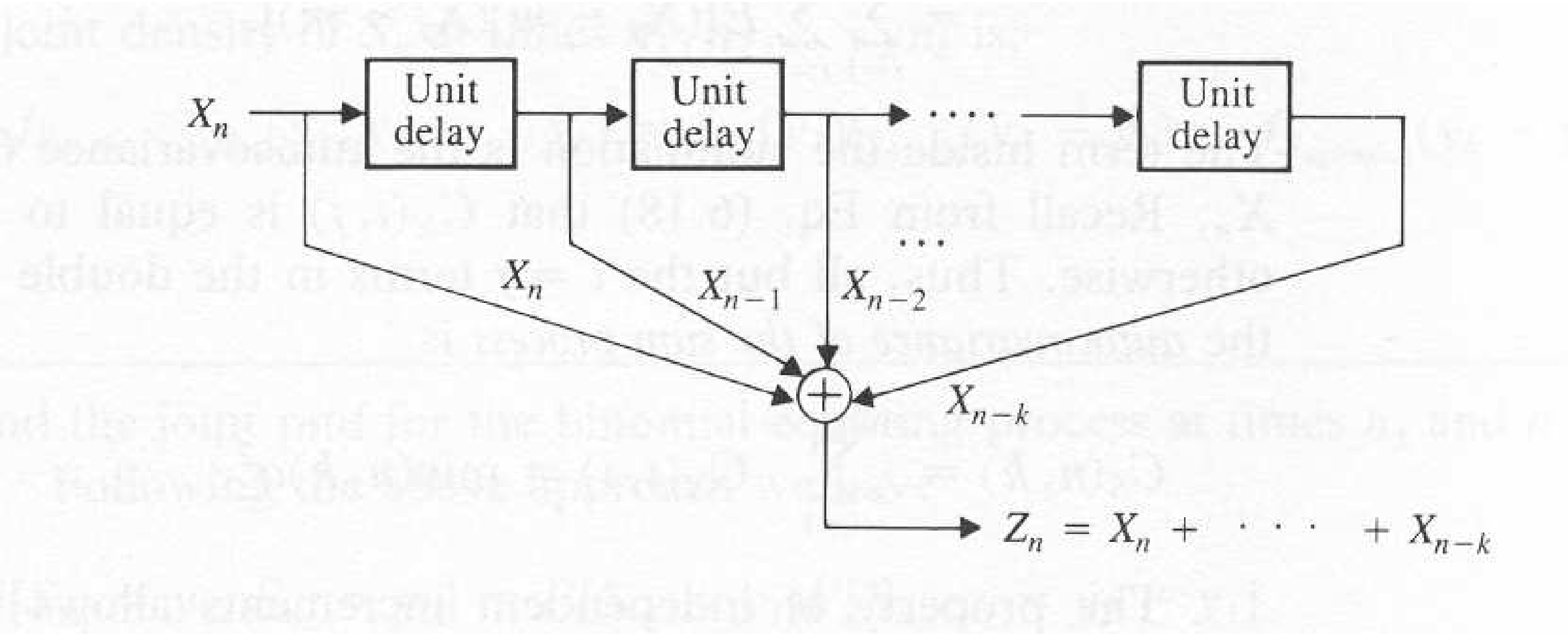
$$C_S(n,k) = E[(S_n - nm)(S_k - km)]$$

$$
\begin{aligned}
&= E[(S_n - nm)\{(S_n - nm) + (S_k - km) - (S_n - nm)\}] \\
&= E[(S_n - nm)^2] + E[(S_n - nm)(S_k - S_n - (k-n)m)] \\
&= E[(S_n - nm)^2] + E[S_n - nm]E[S_k - S_n - (k-n)m] \\
&= E[(S_n - nm)^2] \\
&= \mathrm{VAR}[S_n] = n\sigma^2.
\end{aligned}
$$

first-Order Autoregressive Random Process



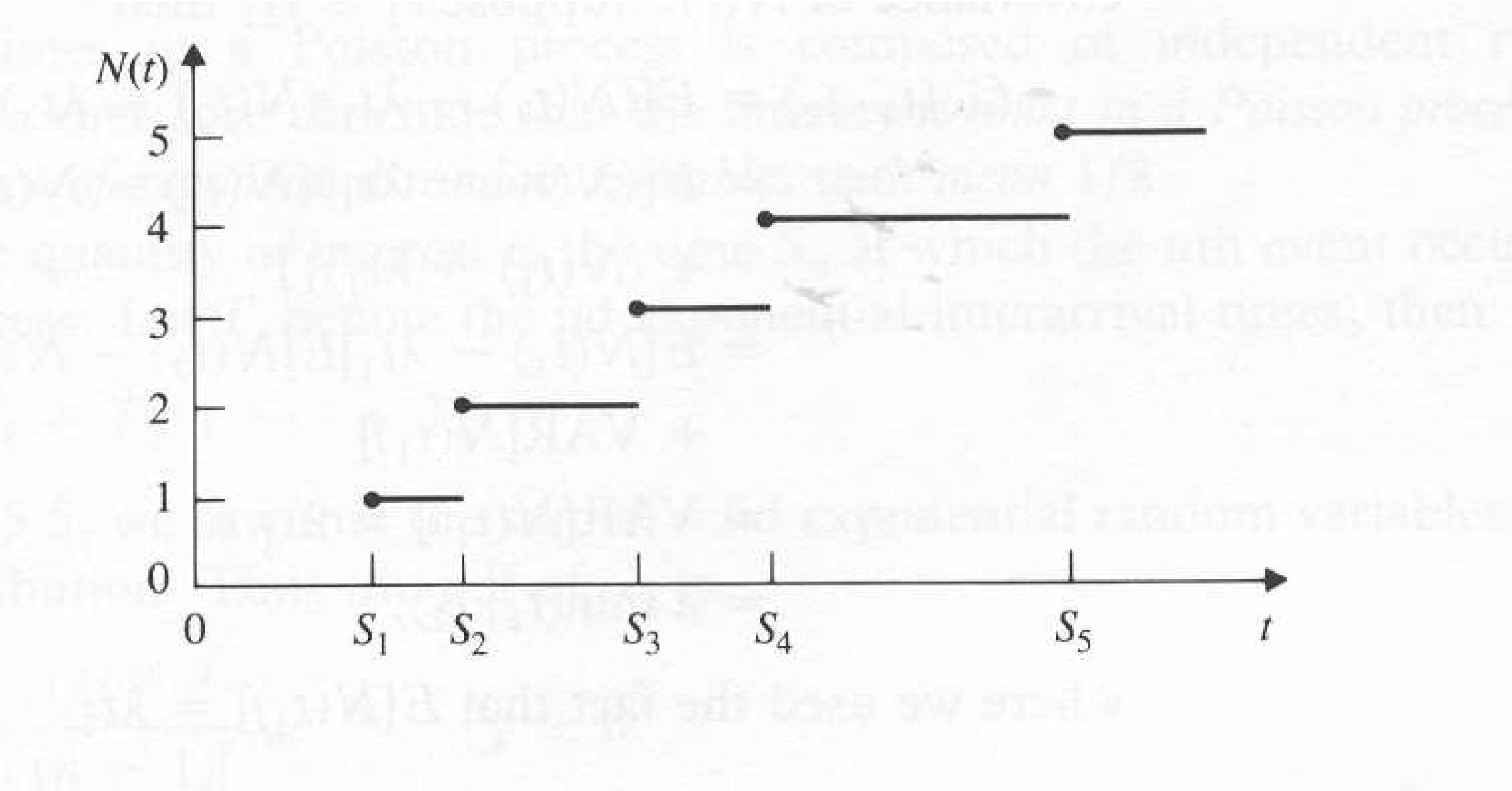
Moving Average Process



6.4 Examples of Continuous-Time Random Processes

Poisson Process

- Events occur at random instants of time at an average rate of λ events per second.
- Let $N(t)$ be the number of event occurrences in the time interval $[0, t]$.



- Divide $[0, t]$ into n subintervals of duration $\delta = t/n$.
- Assume that the following two conditions hold:

1. The probability of more than one event occurrence in a subinterval is negligible compared to the probability of observing one or zero events. – Bernoulli trial
2. Whether or not an event occurs in a subinterval is independent of the outcomes in other subintervals. – Bernoulli trials are independent.

- $N(t)$ can be approximated by the binomial counting process.
- Let p be the prob. of event occurrence in each subinterval. Then the expected number of event occurrence in the interval $[0, t]$ is np.

- The average number of events in the interval $[0, t]$ is also λt. Thus

$$\lambda t = np.$$

- Let $n \to \infty$ and then $p \to 0$ while $np = \lambda t$ remains fixed.
- Binomial distribution approaches Poisson distribution with parameter λt.

$$P[N(t) = k] = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \quad \text{for } k = 0, 1, \ldots$$

- $N(t)$ is called Poisson process.

- We will show that if n is large and p is small, then for $\alpha = np$,

$$p_k = \binom{n}{k} p^k (1-p)^{n-k} \simeq \frac{\alpha^k}{k!} e^{-\alpha} \qquad k = 0, 1, \ldots.$$

- Consider the probability that no events occur in n trials:

$$p_0 = (1-p)^n = \left(1 - \frac{\alpha}{n}\right)^n \to e^{-\alpha} \quad \text{as } n \to \infty.$$

- Let $q = 1 - p$. Noting that

$$\frac{p_{k+1}}{p_k} = \frac{\binom{n}{k+1} p^{k+1} q^{n-k-1}}{\binom{n}{k} p^k q^{n-k}}$$

$$
\begin{aligned}
&= \frac{(n-k)p}{(k+1)q} = \frac{(1-k/n)\alpha}{(k+1)(1-\alpha/n)} \\
&\rightarrow \frac{\alpha}{k+1} \quad \text{as } n \rightarrow \infty.
\end{aligned}
$$

Thus

$$p_{k+1} = \frac{\alpha}{k+1} p_k \quad \text{for } k = 0, 1, 2, \ldots$$

and

$$p_0 = e^{-\alpha}.$$

- A simple induction argument then shows that

$$p_k = \frac{\alpha^k}{k!} e^{-\alpha} \quad \text{for } k = 0, 1, 2, \ldots.$$

- The mean (the variance) of $N(t)$ is λt.

- Independent and stationary increments $\rightarrow$

$$\begin{aligned} P[N(t_1) = i, N(t_2) = j] &= P[N(t_1) = i]P[N(t_2) - N(t_1) = j - i] \\ &= P[N(t_1) = i]P[N(t_2 - t_1) = j - i] \\ &= \frac{(\lambda t_1)^i e^{-\lambda t_1}}{i!} \frac{(\lambda(t_2 - t_1))^{j-i} e^{-\lambda(t_2 - t_1)}}{(j - i)!}. \end{aligned}$$

- Covariance of $N(t)$. Suppose $t_1 \leq t_2$, then

$$\begin{aligned} C_N(t_1, t_2) &= E[(N(t_1) - \lambda t_1)(N(t_2) - \lambda t_2)] \\ &= E[(N(t_1) - \lambda t_1)\{N(t_2) - N(t_1) - \lambda t_2 + \lambda t_1 \\ &\quad +(N(t_1) - \lambda t_1)\}] \\ &= E[N(t_1) - \lambda t_1]E[N(t_2) - N(t_1) - \lambda(t_2 - t_1)] \\ &\quad +\text{VAR}[N(t_1)] \\ &= \text{VAR}[N(t_1)] = \lambda t_1 = \lambda \min(t_1, t_2). \end{aligned}$$

Interevent Times

- Consider the time T between event occurrences in a Poisson process.
- $[0, t]$ is divided into n subintervals of length $\delta = t/n$.
- The probability that $T > t$ is

$$\begin{aligned} P[T > t] &= P[\text{no events in } t \text{ seconds}] \\ &= (1-p)^n \\ &= \left(1 - \frac{\lambda t}{n}\right)^n \\ &\to e^{-\lambda t} \quad \text{as } n \to \infty. \end{aligned}$$

- The cdf of T is then
$$1 - e^{-\lambda t}.$$
- T is an exponential random variable with parameter λ.
- Since the times between event occurrences in the underlying binomial process are independent geometric random variable, it follows that the interevent times in a Poisson Process form an iid sequence of exponential random variables with mean $1/\lambda$.

Individual Arrival Times

- In applications where the Poisson process models customer interarrival times, it is customary to say that arrivals occur "at random."
- Suppose that we are given that only one arrival occurred in an interval $[0, t]$, and let X be the arrival time of the single customer.
- For $0 < x < t$, let $N(x)$ be the number of events up to time x, and let $N(t) - N(x)$ be the increment in the interval $(x, t]$, then

$$P[X \leq x] = P[N(x) = 1 \mid N(t) = 1]$$

$$
\begin{aligned}
&= \frac{P[N(x) = 1 \text{ and } N(t) = 1]}{P[N(t) = 1]} \\
&= \frac{P[N(x) = 1 \text{ and } N(t) - N(x) = 0]}{P[N(t) = 1]} \\
&= \frac{P[N(x) = 1]P[N(t) - N(x) = 0]}{P[N(t) = 1]} \\
&= \frac{\lambda x e^{-\lambda x} e^{-\lambda(t-x)}}{\lambda t e^{-\lambda t}} \\
&= \frac{x}{t}.
\end{aligned}
$$

- It can be shown that *if the number of arrivals in the interval* $[0, t]$ *is* k*, then the individual arrival times are distributed independently and uniformly in the interval.*

6.5 Stationary Random Processes

- We now consider those random processes that Randomness in the processes does not change with time, that is, they have the same behaviors between an observation in (t_0, t_1) and $(t_0 + \tau, t_1 + \tau)$.
- A discrete-time or continuous-time random process $X(t)$ is **stationary** if the joint distribution of any set of samples does not depend on the placement of the time origin. That is,

$$F_{X(t_1),\ldots,X(t_k)}(x_1, \ldots, x_k) = F_{X(t_1+\tau),\ldots,X(t_k+\tau)}(x_1, \ldots, x_k)$$

for all time shift τ, all k, and all choices of sample times $t_1, \ldots, t_k$.

- Two processes $X(t)$ and $Y(t)$ are said to be **jointly stationary** if the joint cdf's of $X(t_1), \ldots, X(t_k)$ and $Y(t'_1), \ldots, Y(t'_j)$ do not depend on the placement of the time origin for all k and j and all choices of sampling times $t_1, \ldots, t_k$ and $t'_1, \ldots, t'_j$.
- The first-order cdf of a stationary random process must be independent of time, i.e.,

$$F_{X(t)}(x) = F_{X(t+\tau)}(x) = F_X(x) \quad \text{for all } t \text{ and } \tau;$$
$$m_X(t) = E[X(t)] = m \quad \text{for all } t;$$
$$\text{VAR}[X(t)] = E[(X(t) - m)^2] = \sigma^2 \quad \text{for all } t.$$

- The second-order cdf of a stationary random process is with

$$F_{X(t_1),X(t_2)}(x_1, x_2) = F_{X(0),X(t_2-t_1)}(x_1, x_2) \quad \text{for all } t_1,\ t_2.$$

- The autocorrelation and autocovariance of stationary random process $X(t)$ depend only on $t_2 - t_1$:

$$R_X(t_1, t_2) = R_X(t_2 - t_1) \quad \text{for all } t_1, t_2;$$
$$C_X(t_1, t_2) = C_X(t_2 - t_1) \quad \text{for all } t_1, t_2.$$

Example: Is the sum process S_n a discrete-time stationary process? We have

$$S_n = X_1 + X_2 + \cdots + X_n,$$

where X_i is an iid sequence.

Since

$$m_S(n) = nm \;\text{ and } \text{VAR}[S_n] = n\sigma^2,$$

mean and variance of S_n are not constant. Thus, S_n cannot be a stationary process.

Wide-Sense Stationary Random Processes

- A discrete-time or continuous-time random process $X(t)$ is **wide-sense stationary** (WSS) if it satisfies

$$m_X(t) = m \quad \text{for all } t \text{ and}$$
$$C_X(t_1, t_2) = C_X(t_1 - t_2) \quad \text{for all } t_1 \text{ and } t_2.$$

- Two processes $X(t)$ and $Y(t)$ are said to be **jointly wide-sense stationary** if they are both wide-sense stationary and if their cross-covariance depends only on $t_1 - t_2$.
- When $X(t)$ is wide-sense stationary, we have

$$C_X(t_1, t_2) = C_X(\tau) \text{ and } R_X(t_1, t_2) = R_X(\tau),$$

where $\tau = t_1 - t_2$.

- Stationary random process $\rightarrow$ wide-sense stationary process

- Assume that $X(t)$ is a wide-sense stationary process.
- The **average power** of $X(t)$ is given by

$$E[X(t)^2] = R_X(0) \quad \text{for all } t.$$

- The autocorrelation function of $X(t)$ is an even function since

$$R_X(\tau) = E[X(t+\tau)X(t)] = E[X(t)X(t+\tau)] = R_X(-\tau).$$

- The autocorrelation function is a measure of the rate of change of a random process.
- Consider the change in the process from time t to $t+\tau$:

$$P[\mid X(t+\tau) - X(t) \mid > \epsilon] \;\; = \;\; P[(X(t+\tau) - X(t))^2 > \epsilon^2]$$

$$\leq \frac{E[(X(t+\tau) - X(t))^2]}{\epsilon^2}$$
$$= \frac{2(R_X(0) - R_X(\tau))}{\epsilon^2},$$

where we apply the Markov inequality to obtain the upper bound. If $R_X(0) - R_X(\tau)$ is small, then the probability of a large change in $X(t)$ in τ seconds is small.

- The autocorrelation function is maximum at $\tau = 0$ since

$$R_X(\tau)^2 = E[X(t+\tau)X(t)]^2 \leq E[X^2(t+\tau)]E[X^2(t)] = R_X(0)^2,$$

where we use the fact that

$$E[XY]^2 \leq E[X^2]E[Y^2].$$

- If $R_X(0) = R_X(d)$, then $R_X(\tau)$ is periodic with period d and $X(t)$ is **mean square periodic**, i.e., $E[(X(t+d) - X(t))^2] = 0$.
- $$\begin{aligned} & E[(X(t+\tau+d) - X(t+\tau))X(t)]^2 \\ & \leq E[(X(t+\tau+d) - X(t+\tau))^2]E[X^2(t)], \end{aligned}$$

  which implies that

$$\{R_X(\tau+d) - R_X(\tau)\}^2 \leq 2\{R_X(0) - R_X(d)\} R_X(0).$$

Therefore,

$$R_X(\tau + d) = R_X(\tau).$$

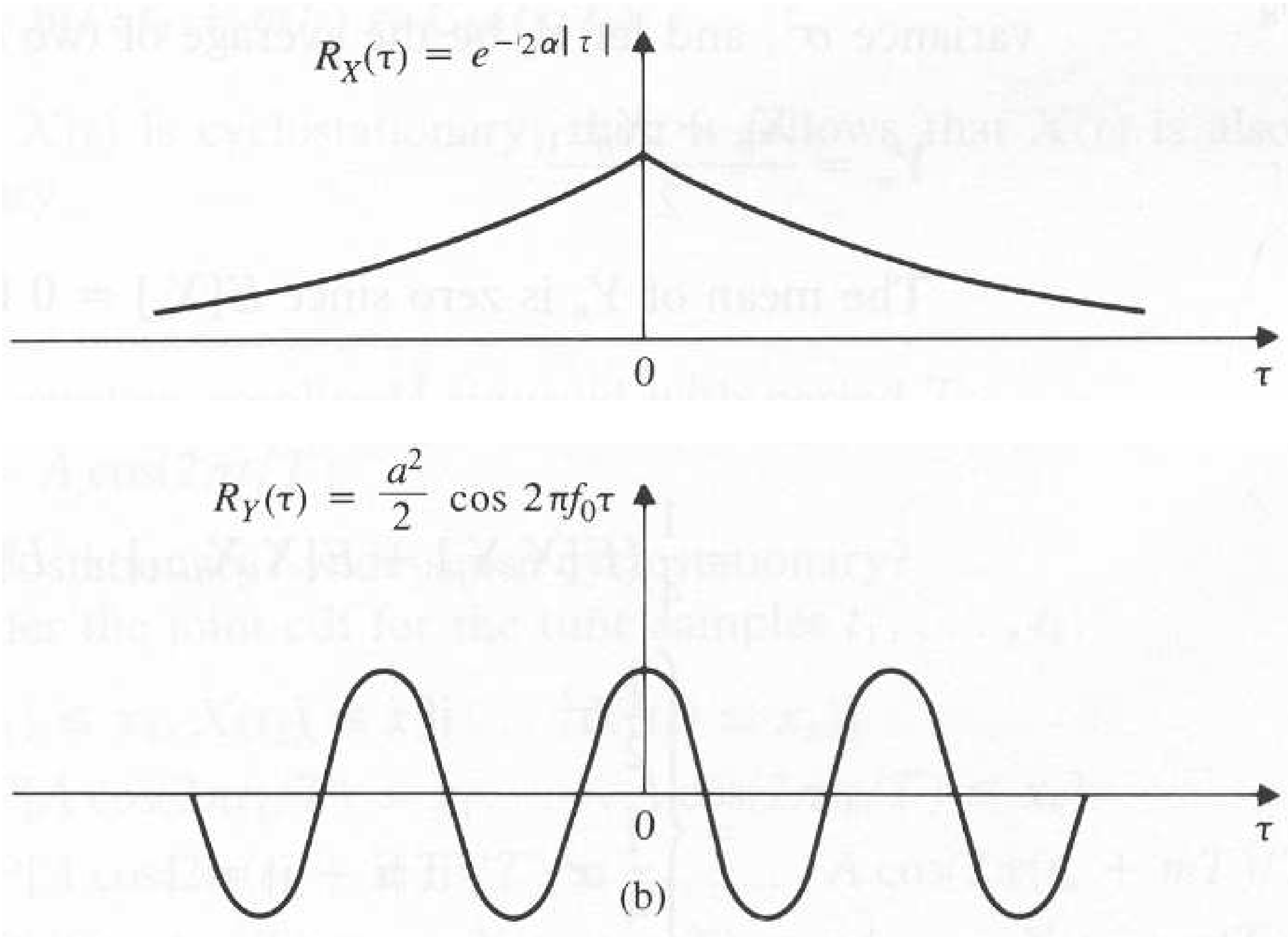
The fact that $X(t)$ is mean square periodic is from

$$E[(X(t+d) - X(t))^2] = 2\{R_X(0) - R_X(d)\} = 0.$$

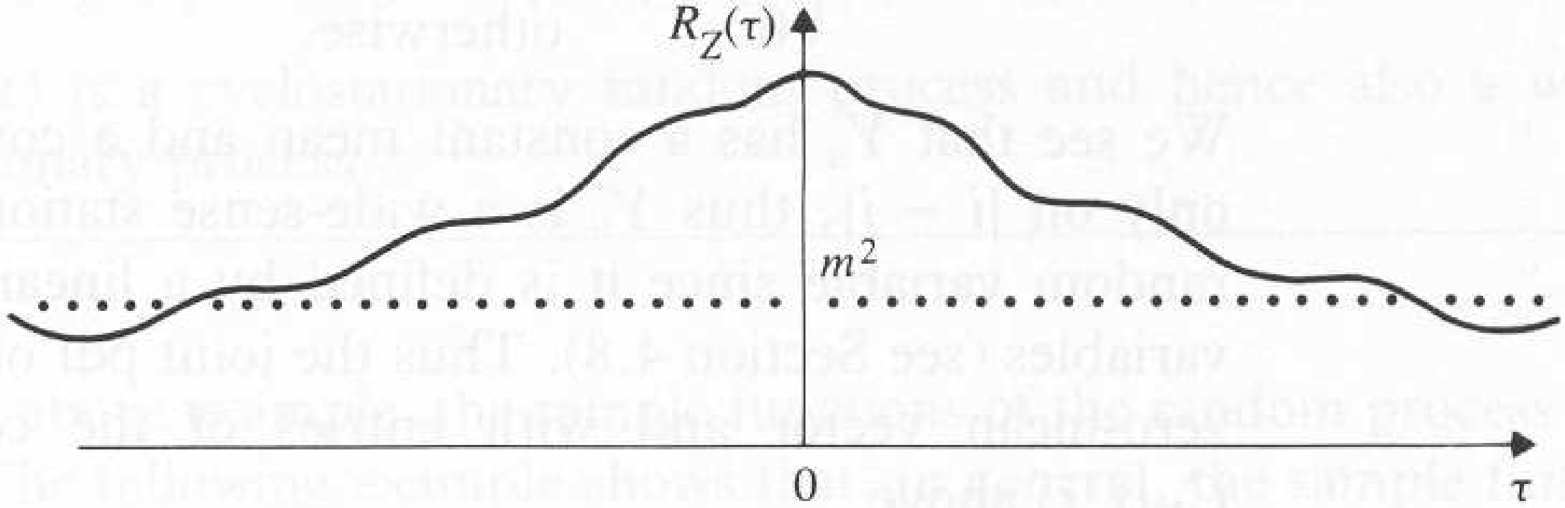
- Let $X(t) = m + N(t)$, where $N(t)$ is a zero-mean process for which $R_N(\tau) \to 0$ as $\tau \to \infty$. Then

$$\begin{aligned} R_X(\tau) &= E[(m + N(t+\tau))(m + N(t))] \\ &= m^2 + 2mE[N(t)] + R_N(\tau) \\ &= m^2 + R_N(\tau) \to m^2 \text{ as } \tau \to \infty. \end{aligned}$$

- In summary, the autocorrelation function can have three types of components: (1) a component that approaches zero as $\tau \to \infty$; (2) a periodic component; and (3) a component due to a non zero mean.



(b)


$R_Z(\tau)$
m^2
0
τ

Wide-Sense Stationary Gaussian Random Processes

- If a Gaussian random process is wide-sense stationary, then it is also stationary.
- This is due to the fact that the joint pdf of a Gaussian random process is completely determined by the mean $m_X(t)$ and the autocovariance $C_X(t_1, t_2)$.

Example: Let X_n be an iid sequence of Gaussian random variables with zero mean and variance σ^2, and let Y_n be

$$Y_n = \frac{X_n + X_{n-1}}{2}.$$

The mean of Y_n is zero since $E[X_i] = 0$ for all i. The covariance of Y_n is

$$
\begin{aligned}
& C_Y(i,j) \\
= \; & E[Y_i Y_j] = \frac{1}{4} E[(X_i + X_{i-1})(X_j + X_{j-1})] \\
= \; & \frac{1}{4}\{E[X_i X_j] + E[X_i X_{j-1}] + E[X_{i-1} X_j] + E[X_{i-1} X_{j-1}]\} \\
= \; & \begin{cases} \frac{1}{2}\sigma^2, & \text{if } i = j \\ \frac{1}{4}\sigma^2, & \text{if } |i-j| = 1 \\ 0, & \text{otherwise} \end{cases}.
\end{aligned}
$$

Y_n is a wide sense stationary process since it has a constant mean and a covariance function that depends only on $|i-j|$.

Y_n is a Gaussian random variable since it is defined by a linear function of Gaussian random variables.

Cyclostationary Random Processes

- A random process $X(t)$ is said to be **cyclostationary** with period T if the joint cdf's of $X(t_1), X(t_2), \ldots, X(t_k)$ and $X(t_1 + mT), X(t_2 + mT), \ldots, X(t_k + mT)$ are the same for all k, m and all choices of $t_1, \ldots, t_k$:

$$F_{X(t_1),X(t_2),\ldots,X(t_k)}(x_1, x_2, \ldots, x_k)$$
$$= F_{X(t_1+mT),X(t_2+mT),\ldots,X(t_k+mT)}(x_1, x_2, \ldots, x_k).$$

- $X(t)$ is said to be **wide-sense cyclostationary** if

$$m_X(t + mT) = m_X(t) \text{ and}$$
$$C_X(t_1 + mT, t_2 + mT) = C_X(t_1, t_2).$$

Example: Consider a random amplitude sinusoid with period T:

$$X(t) = A\cos(2\pi t/T).$$

Is $X(t)$ cyclostationary? wide-sense cyclostationary?

We have

$$\begin{aligned}
& P[X(t_1) \le x_1, X(t_2) \le x_2, \ldots, X(t_k) \le x_k] \\
&= P[A\cos(2\pi t_1/T) \le x_1, \ldots, A\cos(2\pi t_k/T) \le x_k] \\
&= P[A\cos(2\pi(t_1 + mT)/T) \le x_1, \ldots, A\cos(2\pi(t_k + mT)/T) \le x_k] \\
&= P[X(t_1 + mT) \le x_1, X(t_2 + mT) \le x_2, \ldots, X(t_k + mT) \le x_k].
\end{aligned}$$

Thus, $X(t)$ is a cyclostationary random process.

6.7 Time Averages of Random Processes and Ergodic Theorems

- We consider the measurement of repeated random experiments.
- We want to take arithmetic average of the quantities of interest.
- To estimate the mean $m_X(t)$ of a random process $X(t,\zeta)$ we have

$$\hat{m}_X(t) = \frac{1}{N}\sum_{i=1}^{N} X(t,\zeta_i),$$

where N is the number of repetitions of the experiment.

- **Time average** of a single realization is given by

$$\langle X(t) \rangle_T = \frac{1}{2T} \int_{-T}^{T} X(t, \zeta) dt.$$

- **Ergodic theorem** states conditions under which a time average converges as the observation interval becomes large.

- We are interested in ergodic theorems that state when time average converge to the ensemble average.

- The strong law of large numbers given as

$$P\left[\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}X_i = m\right] = 1$$

is one of the most important ergodic theorems, where X_n is an iid discrete-time random process with finite mean $E[X_i] = m$.

Example: Let $X(t) = A$ for all t, where A is a zero mean, unit-variance random variable. Find the limit value of the time average.

The mean of the process $m_X(t) = E[X(t)] = E[A] = 0$. The time average gives

$$\langle X(t) \rangle_T = \frac{1}{2T} \int_{-T}^{T} A dt = A.$$

The time average does not converge to $m_X(t) = 0$. $\rightarrow$ Stationary processes need not be ergodic.

- Let $X(t)$ be a WSS process. Then

$$\begin{aligned} E[\langle X(t)\rangle_T] &= E\left[\frac{1}{2T}\int_{-T}^{T} X(t)dt\right] \\ &= \frac{1}{2T}\int_{-T}^{T} E[X(t)]dt = m. \end{aligned}$$

Hence, $\langle X(t)\rangle_T$ is an unbiased estimator for m.

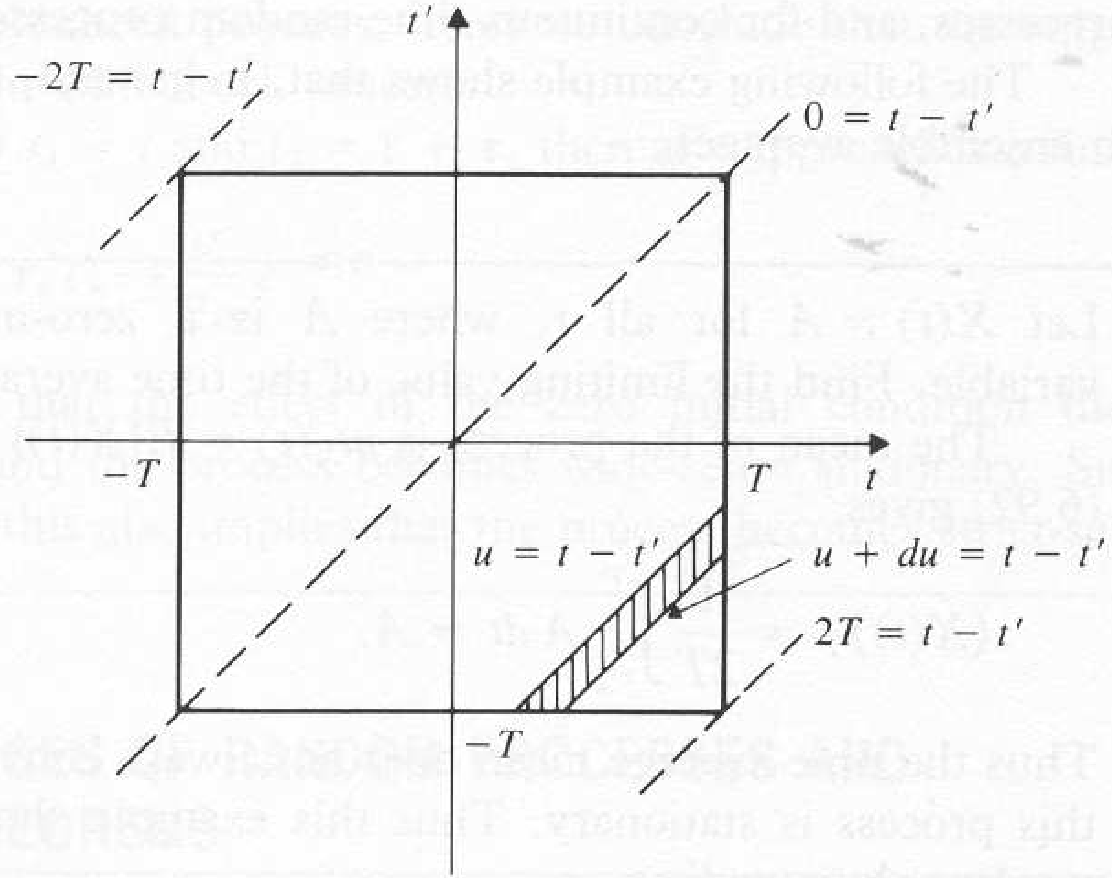
- Variance of $\langle X(t)\rangle_T$ is given by

$$\begin{aligned} &\mathrm{VAR}[\langle X(t)\rangle_T] = E[(\langle X(t)\rangle_T - m)^2] \\ &= E\left[\left\{\frac{1}{2T}\int_{-T}^{T}(X(t)-m)dt\right\}\left\{\frac{1}{2T}\int_{-T}^{T}(X(t')-m)dt'\right\}\right] \\ &= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} E[(X(t)-m)(X(t')-m)]dtdt' \end{aligned}$$

$$= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C_X(t,t')dtdt'.$$

- Since the process $X(t)$ is WSS, we have

$$\begin{aligned} \mathrm{VAR}[\langle X(t)\rangle_T] &= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C_X(t-t')dtdt' \\ &= \frac{1}{4T^2}\int_{-2T}^{2T}(2T-|u|)C_X(u)du \\ &= \frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|u|}{2T}\right)C_X(u)du. \end{aligned}$$


t'
−2T = t − t'
0 = t − t'
−T
T
t
u = t − t'
u + du = t − t'
2T = t − t'
−T

- $\langle X(t)\rangle_T$ will approach m in the mean square sense, that is, $E[(\langle X(t)\rangle_T - m)^2] \to 0$, if $\mathrm{VAR}[\langle X(t)\rangle_T]$ approaches zero.

Theorem: Let $X(t)$ be a WSS process with $m_X(t) = m$, then

$$\lim_{T\to\infty} \langle X(t)\rangle_T = m$$

in the mean square sense, if and only if

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|u|}{2T}\right) C_X(u)du = 0.$$

- Time-average estimate for the autocorrelation function is given by

$$\langle X(t+\tau)X(t)\rangle_T = \frac{1}{2T}\int_{-T}^{T} X(t+\tau)X(t)dt.$$

- $E[\langle X(t+\tau)X(t)\rangle_T] = R_X(\tau)$ if $X(t)$ is WSS random process.
- Time average autocorrelation converges to $R_X(\tau)$ in the mean square sense if $\text{VAR}[\langle X(t+\tau)X(t)\rangle_T]$ converges to zero.

- If the random process is discrete time, then

$$\langle X_n \rangle_T = \frac{1}{2T+1} \sum_{n=-T}^{T} X_n;$$

$$\langle X_{n+k} X_n \rangle_T = \frac{1}{2T+1} \sum_{n=-T}^{T} X_{n+k} X_n.$$

- If X_n is a WSS random process, then $E[\langle X_n \rangle_T] = m$.
- Variance of $\langle X_n \rangle_T$ is given by

$$\text{VAR}[\langle X_n \rangle_T] = \frac{1}{2T+1} \sum_{k=-2T}^{2T} \left(1 - \frac{|k|}{2T+1}\right) C_X(k).$$

- $\langle X_n \rangle_T$ approaches m in the mean square sense and is mean ergodic if

$$\frac{1}{2T+1} \sum_{k=-2T}^{2T} \left(1 - \frac{|k|}{2T+1}\right) C_X(k) \to 0.$$