# Chapter 7: Analysis and Processing of Random Signals[1]

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[1] Modified from the lecture notes by Prof. Mao-Ching Chiu

# 7.1 Power Spectral Density

- Fourier series and Fourier transform – Analysis of nonrandom time function in the frequency domain.
- For WSS processes $X(t)$, the autocorrelation function $R_X(\tau)$ is an measure for the average rate of change of $X(t)$.
- **Einstein-Wiener-Khinchin Theorem**: Power spectral density of a WSS random process is given by the Fourier transform of the autocorrelation function.

## Continuous-Time Random Process

- $X(t)$ is a continuous-time WSS random process with mean $m_X$ and autocorrelation function $R_X(\tau)$.
- The **power-spectral density** of $X(t)$ is given by the Fourier transform of the autocorrelation function.

$$\begin{aligned} S_X(f) &= \mathcal{F}\{R_X(\tau)\} \\ &= \int_{-\infty}^{+\infty} R_X(\tau)e^{-j2\pi f\tau}\,d\tau. \end{aligned}$$

- If $X(t)$ is real value, then

$$R_X(\tau) = R_X(-\tau).$$

We have

$$
\begin{aligned}
S_X(f) &= \int_{-\infty}^{+\infty} R_X(\tau)[\cos(2\pi f\tau) + j\sin(2\pi f\tau)]d\tau \\
&= \int_{-\infty}^{+\infty} R_X(\tau)\cos(2\pi f\tau)d\tau.
\end{aligned}
$$

- Inverse Fourier transform is given by

$$
\begin{aligned}
R_X(\tau) &= \mathcal{F}^{-1}\{S_X(f)\} \\
&= \int_{-\infty}^{+\infty} S_X(f)e^{j2\pi f\tau}df.
\end{aligned}
$$

- Average power of $X(t)$ is

$$
E[X^2(t)] = R_X(0) = \int_{-\infty}^{+\infty} S_X(f)df.
$$

- $S_X(f)$ is the *density of power* of $X(t)$ at the frequency $f$.
- Since $R_X(\tau) = C_X(\tau) + m_X^2$, the power spectral density is also given by

$$\begin{aligned} S_X(f) &= \mathcal{F}\{C_X(\tau) + m_X^2\} \\ &= \mathcal{F}\{C_X(\tau)\} + m_X^2 \delta(f). \end{aligned}$$

  Note that $m_X^2$ is the "dc" component of $X(t)$.
- **Cross-power spectral density** $S_{X,Y}(f)$ is defined by

$$S_{X,Y}(f) = \mathcal{F}\{R_{X,Y}(\tau)\},$$

  where

$$R_{X,Y}(\tau) = E[X(t+\tau)Y(t)].$$

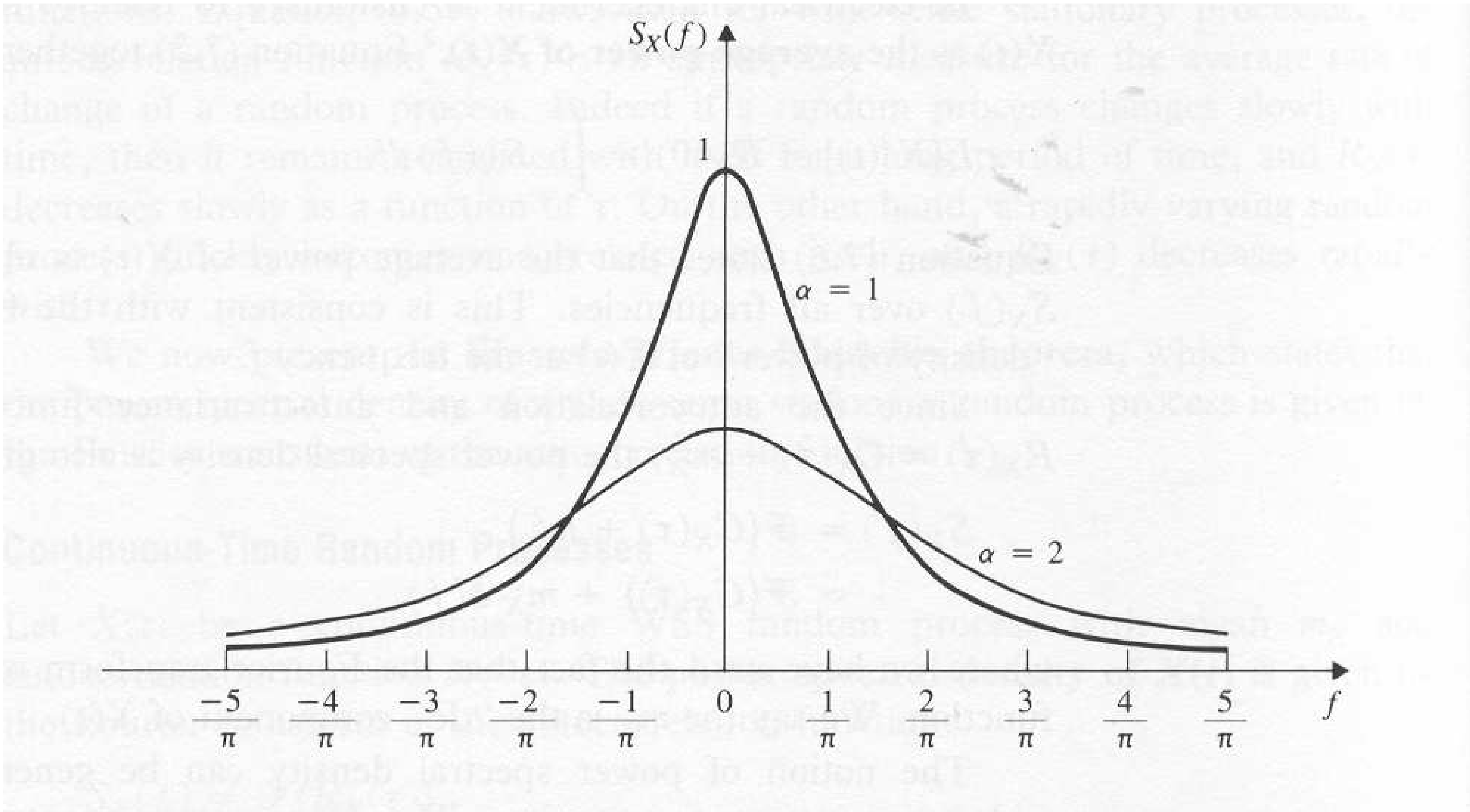
- In general, $S_{X,Y}(f)$ is a complex function of $f$.

**Example**: The autocorrelation function of the random telegraph process is given by

$$R_X(\tau) = e^{-2\alpha|\tau|}.$$

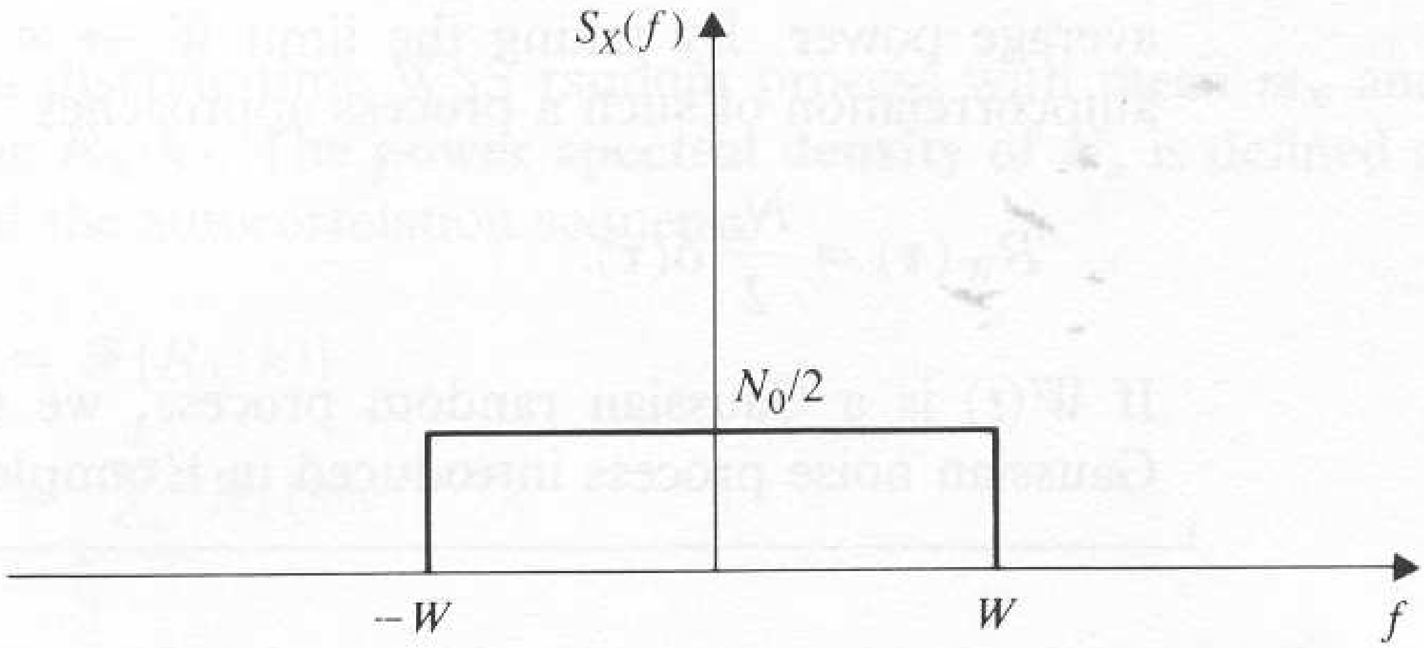
The power spectral density is

$$\begin{aligned} S_X(f) &= \int_{-\infty}^{0} e^{2\alpha\tau} e^{-j2\pi f\tau} d\tau + \int_{0}^{\infty} e^{-2\alpha\tau} e^{-j2\pi f\tau} \\ &= \frac{1}{2\alpha - j2\pi f} + \frac{1}{2\alpha + j2\pi f} \\ &= \frac{4\alpha}{4\alpha^2 + 4\pi^2 f^2}. \end{aligned}$$



$S_X(f)$
1
$\alpha = 1$
$\alpha = 2$
$\frac{-5}{\pi}$ $\frac{-4}{\pi}$ $\frac{-3}{\pi}$ $\frac{-2}{\pi}$ $\frac{-1}{\pi}$ 0 $\frac{1}{\pi}$ $\frac{2}{\pi}$ $\frac{3}{\pi}$ $\frac{4}{\pi}$ $\frac{5}{\pi}$
$f$

**Example**: The power spectral density of a WSS white noise whose frequency components are limited to $-W \leq f \leq W$ is shown in the following figure:

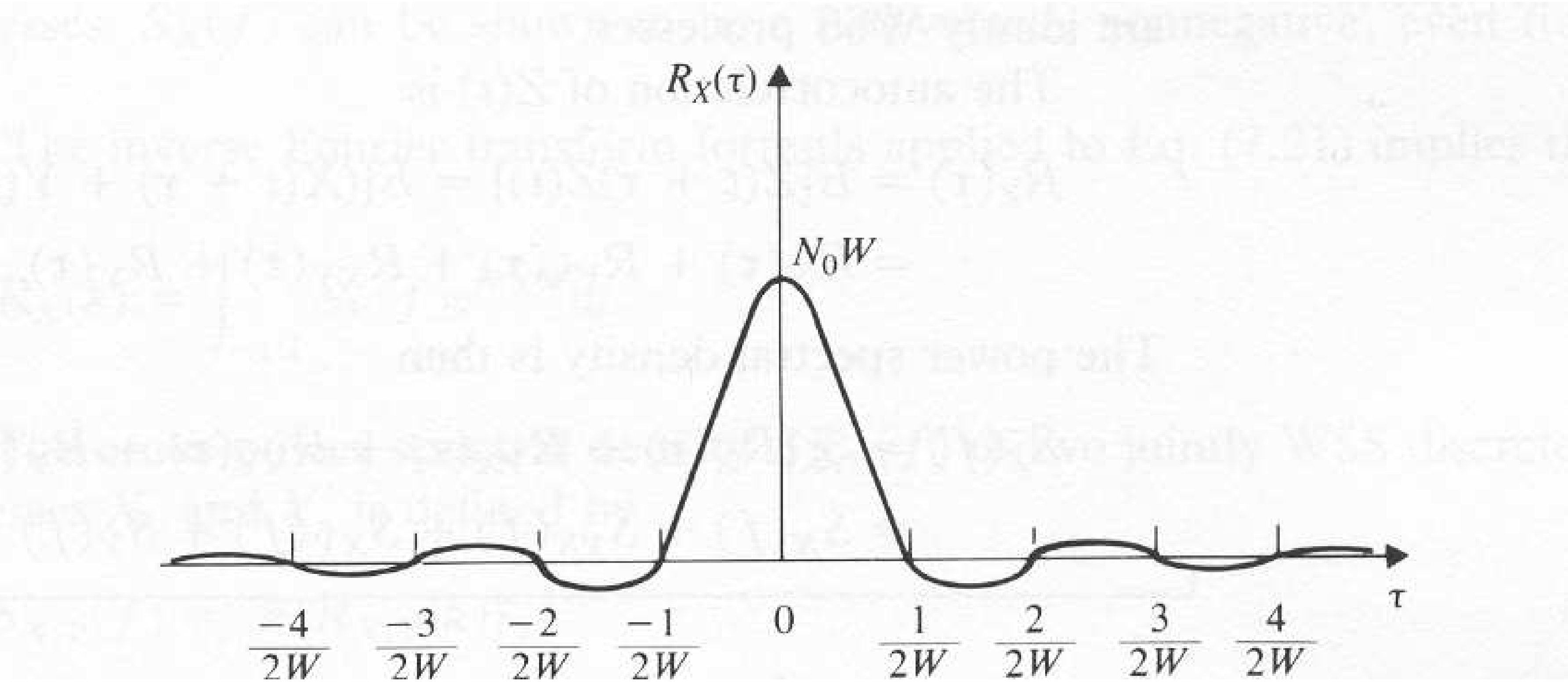


The average power is

$$E[X^2(t)] = \int_{-W}^{W} \frac{N_0}{2} df = N_0 W.$$

The autocorrelation function for this process is

$$\begin{aligned} R_X(\tau) &= \frac{1}{2} N_0 \int_{-W}^{W} e^{j2\pi f\tau} df \\ &= \frac{1}{2} N_0 \frac{e^{-j2\pi W\tau} - e^{j2\pi W\tau}}{-j2\pi\tau} \\ &= \frac{N_0 \sin(2\pi W\tau)}{2\pi\tau}. \end{aligned}$$




- White noise usually refers to a random process $W(t)$

whose power spectral density is $N_0/2$ for all frequencies:

$$S_W(f) = \frac{N_0}{2} \qquad \text{for all } f.$$

- White noise has infinity average power.
- Autocorrelation function of $W(t)$ is

$$R_W(\tau) = \frac{N_0}{2}\delta(\tau).$$

- If $W(t)$ is a Gaussian random process, then $W(t)$ is the white Gaussian noise process.

**Example**: Find the power spectral density of $Z(t) = X(t) + Y(t)$, where $X(t)$ and $Y(t)$ are jointly WSS process. The autocorrelation function of $Z(t)$ is

$$\begin{aligned} R_Z(\tau) &= E[Z(t+\tau)Z(t)] \\ &= E[(X(t+\tau) + Y(t+\tau))(X(t) + Y(t))] \\ &= R_X(\tau) + R_{YX}(\tau) + R_{XY}(\tau) + R_Y(\tau). \end{aligned}$$

The power spectral density is

$$\begin{aligned} S_Z(f) &= \mathcal{F}\{R_X(\tau) + R_{YX}(\tau) + R_{XY}(\tau) + R_Y(\tau)\} \\ &= S_X(f) + S_{YX}(f) + S_{XY}(f) + S_Y(f). \end{aligned}$$

## Discrete-Time Random Process

- Let $X_n$ be a discrete-time WSS random process with mean $m_X$ and autocorrelation function $R_X(k)$.
- The **power spectral density** of $X_n$ is defined as the Fourier transform

$$\begin{aligned} S_X(f) &= \mathcal{F}\{R_X(k)\} \\ &= \sum_{k=-\infty}^{\infty} R_X(k)e^{-j2\pi fk}. \end{aligned}$$

- We only need to consider frequencies in the range $-1/2 < f \leq 1/2$, since $S_X(f)$ is periodic in $f$ with period 1.

- Inverse Fourier transform is given by

$$R_X(k) = \int_{-1/2}^{1/2} S_X(f) e^{j2\pi f k} df.$$

- The **cross-power spectral density** $S_{XY}(f)$ of two joint WSS discrete-time processes $X_n$ and $Y_n$ is defined by

$$S_{X,Y}(f) = \mathcal{F}\{R_{X,Y}(k)\}$$

and

$$R_{X,Y}(k) = E[X_{n+k} Y_n].$$

**Example:** Let the process $X_n$ be a sequence of uncorrelated random variables with zero mean and variance $\sigma_X^2$. Find $S_X(f)$.

$$R_X(k) = \begin{cases} \sigma_X^2 & k = 0 \\ 0 & k \neq 0 \end{cases}.$$

The power spectral density of the process can be found to be

$$S_X(f) = \sigma_X^2 \qquad -\frac{1}{2} < f < \frac{1}{2}.$$

**Example**: Let $Y_n = X_n + \alpha X_{n-1}$, where $X_n$ is the white noise process given in the previous example. Find $S_Y(f)$.

**Sol**: The mean and autocorrelation function of $Y_n$ are given by

$$E[Y_n] = 0$$

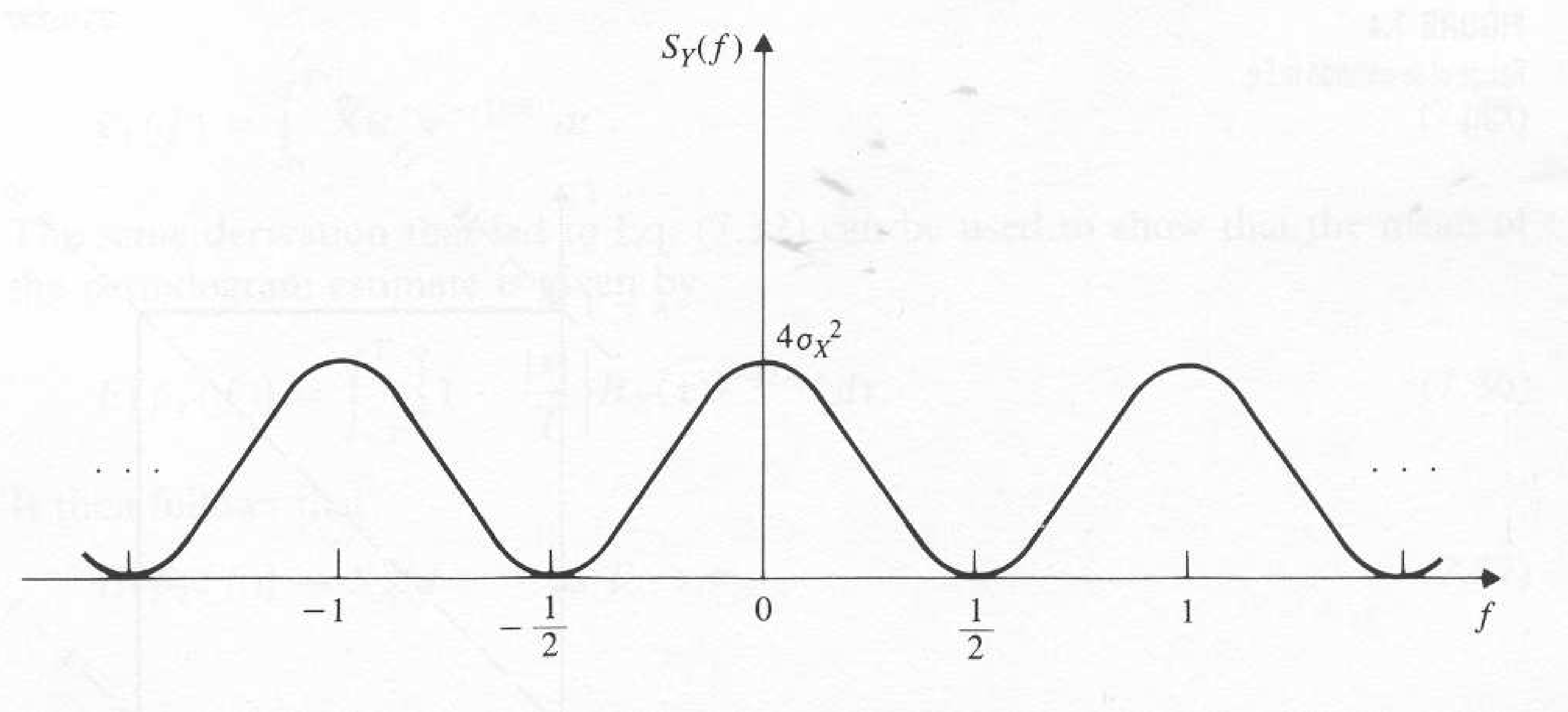
and

$$E[Y_n Y_{n+k}] = \begin{cases} (1+\alpha^2)\sigma_X^2 & k = 0 \\ \alpha\sigma_X^2 & k = \pm 1 \\ 0 & \text{otherwise} \end{cases}.$$

The power spectral density is then

$$\begin{aligned} S_Y(f) &= (1+\alpha^2)\sigma_X^2 + \alpha\sigma_X^2(e^{j2\pi f} + e^{-j2\pi f}) \\ &= \sigma_X^2\{(1+\alpha^2) + 2\alpha\cos(2\pi f)\}. \end{aligned}$$

$\alpha = 1$



**Example**: Let the observation $Z_n$ is given by $Z_n = X_n + Y_n$, where $X_n$ is the signal we wish to observe, $Y_n$ is a white noise process with power $\sigma_Y^2$, and $X_n$ and $Y_n$ are independent. Suppose that $X_n = A$ for all $n$, where $A$ is a random variable with zero mean and variance $\sigma_A^2$. Find the power spectral density of $Z_n$.

**Sol:** The mean and autocorrelation of $Z_n$ are

$$E[Z_n] = E[A] + E[Y_n] = 0$$

and

$$\begin{aligned} E[Z_n Z_{n+k}] &= E[(X_n + Y_n)(X_{n+k} + Y_{n+k})] \\ &= E[X_n X_{n+k}] + E[X_n]E[Y_{n+k}] \end{aligned}$$

$$\begin{aligned} & +E[X_{n+k}]E[Y_n] + E[Y_nY_{n+k}] \\ = \quad & E[A^2] + R_Y(k). \end{aligned}$$

Thus $Z_n$ is also a WSS process. The power spectral density of $Z_n$ is then

$$S_Z(f) = E[A^2]\delta(f) + S_Y(f).$$

## Power Spectral Density as a Time Average

- Let $X_0, \ldots, X_{k-1}$ be $k$ observations from the discrete-time, WSS process $X_n$. The Fourier transform of this sequence is

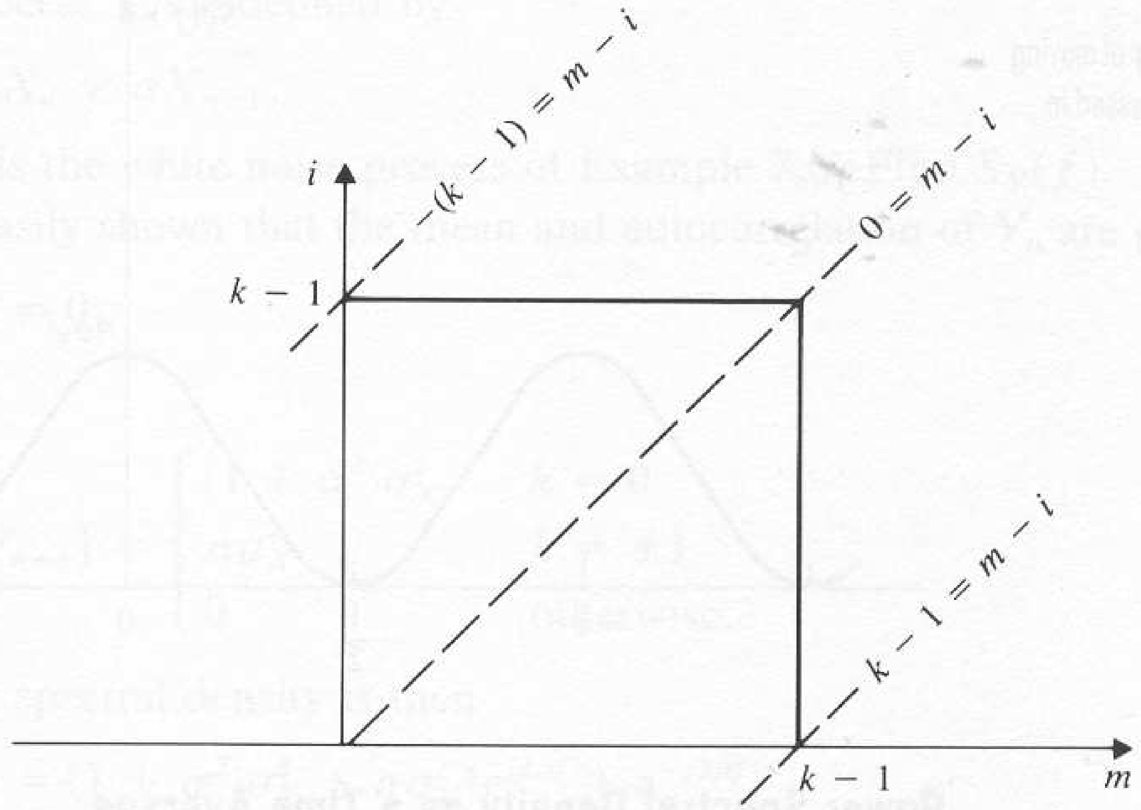
$$\tilde{x}_k(f) = \sum_{m=0}^{k-1} X_m e^{-j2\pi f m}$$

- $|\tilde{x}_k(f)|^2$ is a measure of the "energy" at frequency $f$.
- Divide this energy by total "time" $k$, we obtain an estimate for the power at frequency $f$:

$$\tilde{p}_k(f) = \frac{1}{k}|\tilde{x}_k(f)|^2.$$

- $\tilde{p}_k(f)$ is called the *periodogram estimate*.
- Consider the expected value of the periodogram estimate:

$$\begin{aligned}
E[\tilde{p}_k(f)] &= \frac{1}{k}E[\tilde{x}_k(f)\tilde{x}_k^*(f)] \\
&= \frac{1}{k}E\left[\sum_{m=0}^{k-1} X_m e^{-j2\pi fm} \sum_{i=0}^{k-1} X_i e^{j2\pi fi}\right] \\
&= \frac{1}{k}\sum_{m=0}^{k-1}\sum_{i=0}^{k-1} E[X_m X_i]e^{-j2\pi f(m-i)} \\
&= \frac{1}{k}\sum_{m=0}^{k-1}\sum_{i=0}^{k-1} R_X(m-i)e^{-j2\pi f(m-i)}.
\end{aligned}$$


i
k − 1
−(k − 1) = m − i
0 = m − i
k − 1 = m − i
k − 1
m

By the above figure, we have

$$\begin{aligned} E[\tilde{p}_k(f)] &= \frac{1}{k} \sum_{m'=-(k-1)}^{k-1} \{k - |m'|\} R_X(m') e^{-j2\pi f m'} \\ &= \sum_{m'=-(k-1)}^{k-1} \left\{1 - \frac{|m'|}{k}\right\} R_X(m') e^{-j2\pi f m'}. \end{aligned}$$

As $k \to \infty$, we have

$$E[\tilde{p}_k(f)] \to S_X(f).$$

The above result shows that $S_X(f)$ is nonnegative for all $f$ since $\tilde{p}_k(f)$ is nonnegative for all $f$.

For continuous-time WSS random process $X(t)$, based on the observation in the interval $(0, T)$, we have

$$\tilde{p}_T(f) = \frac{1}{T}|\tilde{x}_T(f)|^2.$$

The result shows

$$\lim_{T\to\infty} E[\tilde{p}_T(f)] = S_X(f).$$

## 7.2 Response of Linear Systems to Random Signals

- Prediction based on previous data
- Filtering and Smoothing
- Modulation

## Continuous-Time Systems

- Consider a system in which an input signal $x(t)$ is mapped into the output signal $y(t)$ by the transformation:

$$y(t) = T[x(t)].$$

- The system is linear if

$$T[\alpha x_1(t) + \beta x_2(t)] = \alpha T[x_1(t)] + \beta T[x_2(t)].$$

- Time-invariant system is given by

$$\begin{aligned} \text{Input } x(t) &\quad\rightarrow\quad \text{Output } y(t); \\ \text{Input } x(t-\tau) &\quad\rightarrow\quad \text{Output } y(t-\tau). \end{aligned}$$

- Impulse response of an LTI system is given by

$$h(t) = T[\delta(t)].$$

- The response of an LTI system to an input $x(t)$ is

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(s)x(t-s)ds = \int_{-\infty}^{+\infty} h(t-s)x(s)ds.$$

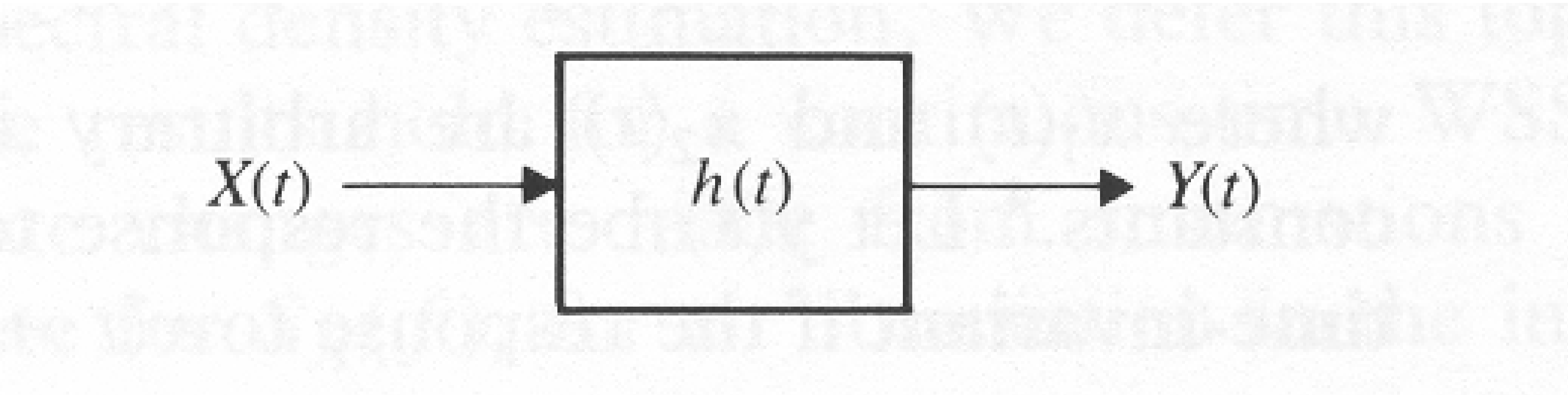
- The transfer function of the system is given by

$$H(f) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{+\infty} h(t)e^{-j2\pi ft}dt.$$

- A system is Causal if the response at time $t$ depends only on past values of the input, that is, if $h(t) = 0$ for $t < 0$.

- If a random process $X(t)$ is the input of an LTI system, then

$$Y(t) = \int_{-\infty}^{+\infty} h(s)X(t-s)ds = \int_{-\infty}^{+\infty} h(t-s)X(s)ds.$$




- If $X(t)$ is WSS, then $Y(t)$ is also WSS.

Proof: The mean of $Y(t)$ is given by

$$\begin{aligned} E[Y(t)] &= E\left[\int_{-\infty}^{+\infty} h(s)X(t-s)ds\right] = \int_{-\infty}^{+\infty} h(s)E[X(t-s)]ds \\ &= m_X \int_{-\infty}^{+\infty} h(\tau)d\tau = m_X H(0). \end{aligned}$$

The auto correlation function is given by

$$\begin{aligned} E[Y(t)Y(t+\tau)] &= E\left[\int_{-\infty}^{+\infty} h(s)X(t-s)ds \int_{-\infty}^{+\infty} h(r)X(t+\tau-r)dr\right] \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(s)h(r)E[X(t-s)X(t+\tau-r)]dsdr \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(s)h(r)R_X(\tau+s-r)dsdr \\ &\rightarrow \text{depends only on } \tau. \end{aligned}$$

## Power Spectral Density of the Output

- Taking the transform of $R_Y(\tau)$ we have

$$\begin{aligned} S_Y(f) &= \int_{-\infty}^{+\infty} R_Y(\tau) e^{-j2\pi f\tau} d\tau \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(s)h(r)R_X(\tau+s-r)e^{-j2\pi f\tau} ds dr d\tau. \end{aligned}$$

Changing variables and letting $u = \tau + s - r$, we have

$$\begin{aligned} S_Y(f) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(s)h(r)R_X(u)e^{-j2\pi f(u-s+r)} ds dr du \\ &= \int_{-\infty}^{+\infty} h(s)e^{j2\pi fs} ds \int_{-\infty}^{+\infty} h(r)e^{-j2\pi fr} dr \int_{-\infty}^{+\infty} R_X(u)e^{-j2\pi fu} du \\ &= H^*(f)H(f)S_X(f) \\ &= |H(f)|^2 S_X(f). \end{aligned}$$

- Mean and autocorrelation function of $Y(t)$ are not sufficient to determine probabilities of events involving $Y(t)$.
- If the input is a Gaussian WSS process, the output is also a Gaussian WSS process which is completely specified by the mean and autocorrelation function of $Y(t)$.
- It can be shown that

$$R_{Y,X}(\tau) = R_X(\tau) * h(\tau);$$

$$S_{Y,X}(\tau) = H(f)S_X(f);$$

$$S_{X,Y}(f) = S^*_{Y,X}(f) = H^*(f)S_X(f).$$

**Example**: Find the power spectral density of the output of a linear, time-invariant system whose input is a white noise process.

**Sol:** Let $X(t)$ be the input process with

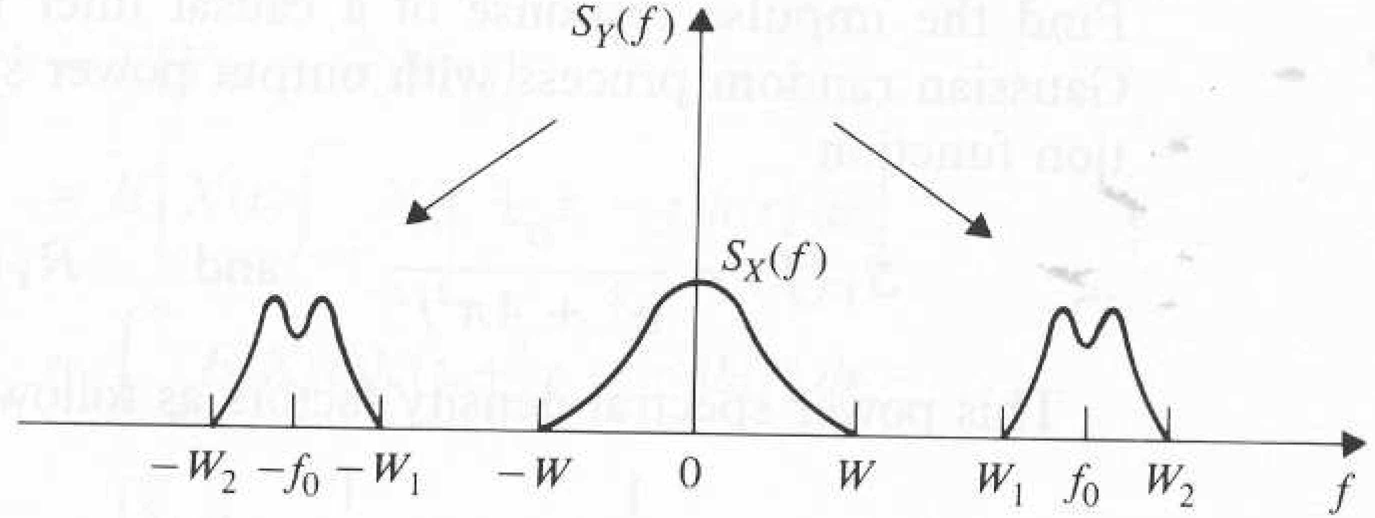
$$S_X(f) = \frac{N_0}{2} \quad \text{for all } f.$$

The power spectral density of the output $Y(t)$ is then

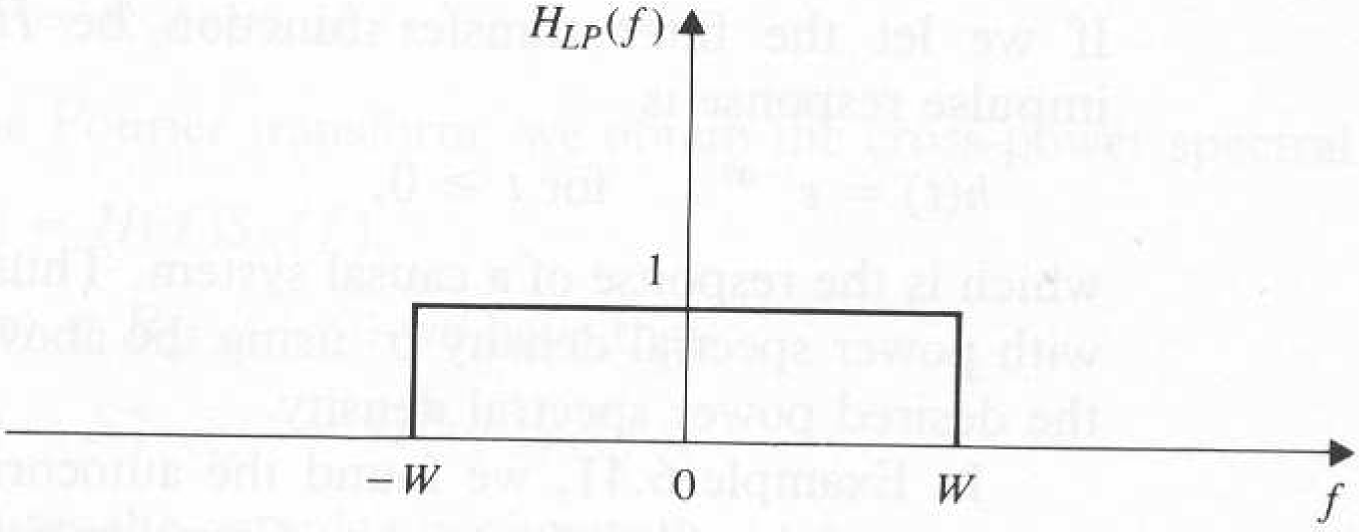
$$S_Y(f) = |H(f)|^2 \frac{N_0}{2}.$$

- One can generate WSS processes with arbitrary power spectral density $S_Y(f)$ by passing a white noise through a system with transfer function $H(f) = \sqrt{S_Y(f)}$.

**Example**: Let $Z(t) = X(t) + Y(t)$




Find the output $W(t)$ if $Z(t)$ is input into an ideal lowpass filter shown below:


$H_{LP}(f)$
1
$-W$
0
$W$
$f$

**Sol:** The power spectral density of the output $W(t)$ is

$$S_W(f) = |H_{LP}(f)|^2 S_X(f) + |H_{LP}(f)|^2 S_Y(f) = S_X(f).$$

Thus, $W(t)$ has the same power spectral density as $X(t)$. This does not imply that $W(t) = X(t)$. To show that $W(t) = X(t)$, in the mean square sense, consider $D(t) = W(t) - X(t)$. Then

$$R_D(\tau) = R_W(\tau) - R_{WX}(\tau) - R_{XW}(\tau) + R_X(\tau).$$

The corresponding power spectral density is

$$\begin{aligned} S_D(f) &= S_W(f) - S_{WX}(f) - S_{XW}(f) + S_X(f) \\ &= |H_{LP}(f)|^2 S_X(f) - H_{LP}(f) S_X(f) - H^*_{LP}(f) S_X(f) + S_X(f) \\ &= 0. \end{aligned}$$

Therefore $R_D(\tau) = 0$ for all $\tau$, and $W(t) = X(t)$ in the mean square sense since

$$E[(W(t) - X(t))^2] = E[D^2(t)] = R_D(0) = 0.$$

## Discrete-Time Systems

- **Unit-sample response** $h_n$ is the response of a discrete-time LTI system to the input

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}.$$

- The response of the system to $X_n$ is given by

$$Y_n = h_n * X_n = \sum_{j=-\infty}^{\infty} h_j X_{n-j} = \sum_{j=-\infty}^{\infty} h_{n-j} X_j.$$

- **Transfer function** of such system is defined by

$$H(f) = \sum_{i=-\infty}^{\infty} h_i e^{-j2\pi f i}.$$

- If $X_n$ is a WSS process, then $Y_n$ is also a WSS process.
- The mean of $Y_n$ is given by

$$m_Y = m_X \sum_{j=-\infty}^{\infty} h_j = m_X H(0).$$

- The autocorrelation of $Y_n$ is given by

$$R_Y(k) = \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h_j h_i R_X(k+j-i)$$

- The power spectral density of $Y_n$ is given by

$$S_Y(f) = |H(f)|^2 S_X(f).$$

**Example**: An autoregressive moving average (ARMA) process is defined by

$$Y_n = -\sum_{i=1}^{q} \alpha_i Y_{n-i} + \sum_{i'=0}^{p} \beta_{i'} W_{n-i'},$$

where $W_n$ is a WSS, white noise input process. The transfer function can be shown to be

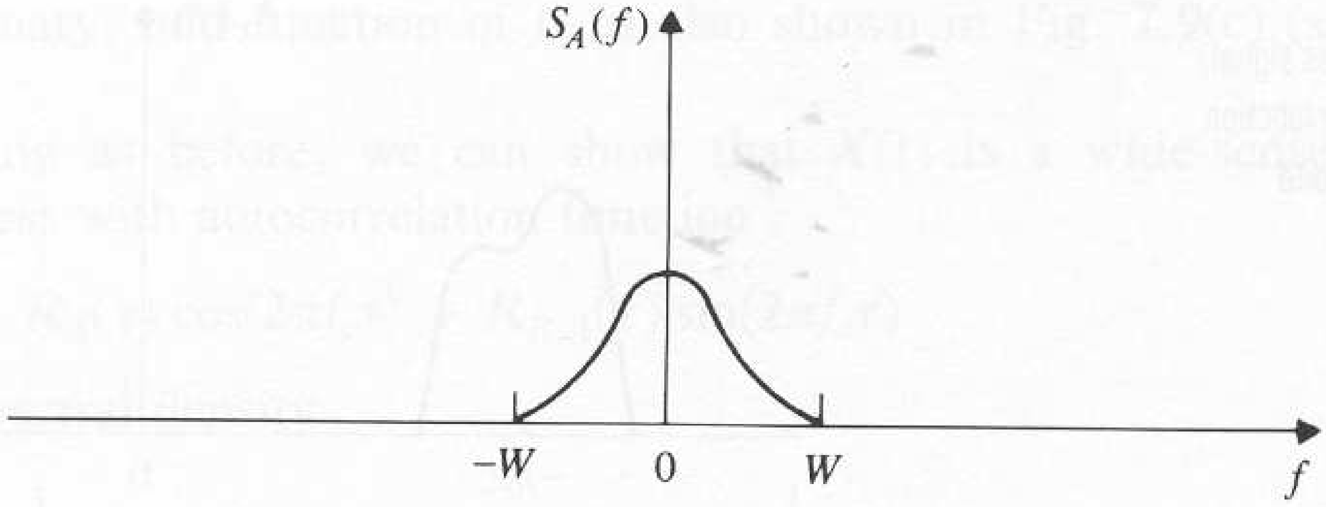
$$H(f) = \frac{\sum_{i'=0}^{p} \beta_{i'} e^{-j2\pi f i'}}{1 + \sum_{i=1}^{q} \alpha_i e^{-j2\pi f i}}.$$

The power spectral density of the ARMA process is

$$S_Y(f) = |H(f)|^2 \sigma_W^2.$$

## 7.3 Amplitude Modulation By Random Signals

- The purpose of a modulator is to map the information signal $A(t)$ into a transmission signal $X(t)$.
- $A(t)$ is a WSS random Process.
- $A(t)$ is a low-pass signal.




(a)

- Amplitude modulation (AM) is given by

$$X(t) = A(t)\cos(2\pi f_c t + \Theta),$$

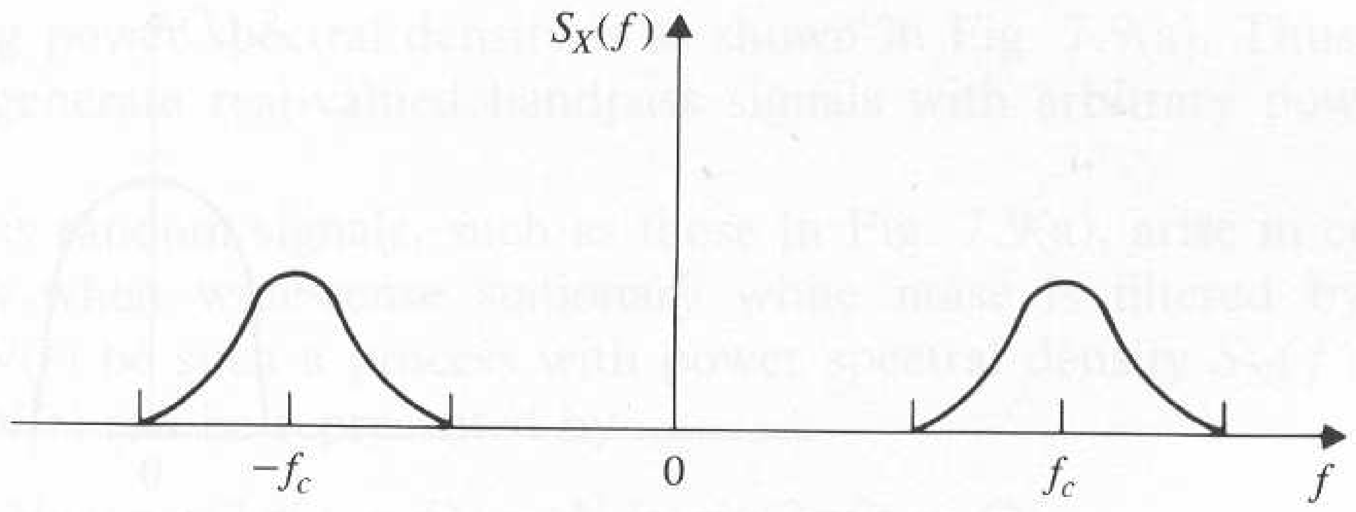
where $\Theta$ is a random variable that is uniformly distributed in the interval $(0, 2\pi)$ and $\Theta$ and $A(t)$ are independent.

- The autocorrelation of $X(t)$ is given by

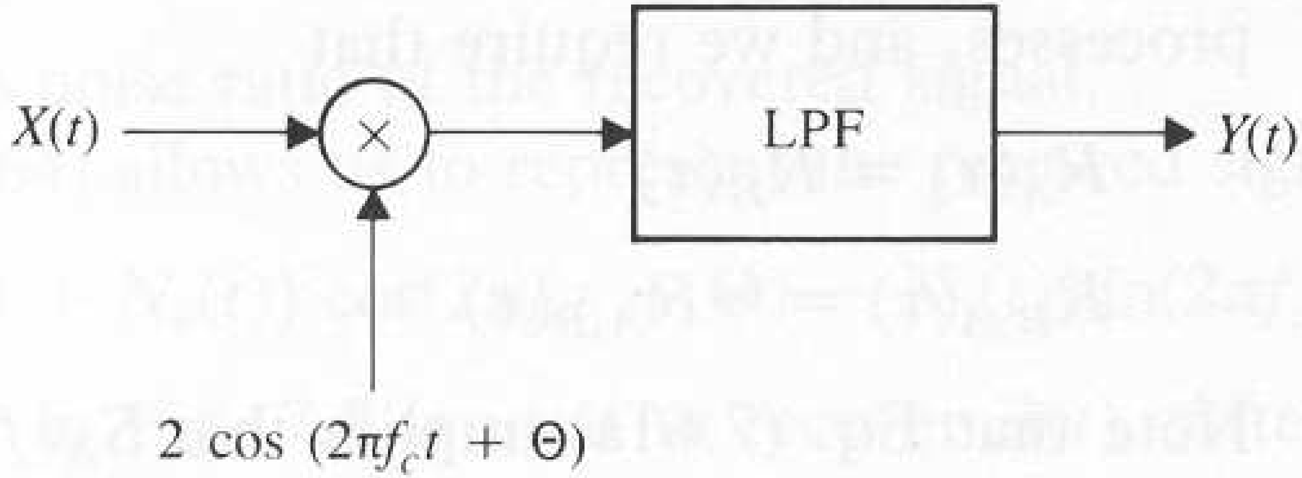
$$\begin{aligned} & E[X(t+\tau)X(t)] \\ &= E[A(t+\tau)\cos(2\pi f_c(t+\tau)+\Theta)A(t)\cos(2\pi f_c t+\Theta)] \\ &= R_A(\tau)E\left[\frac{1}{2}\cos(2\pi f_c t) + \frac{1}{2}\cos(2\pi f_c(2t+\tau)+2\Theta)\right] \\ &= \frac{1}{2}R_A(\tau)\cos(2\pi f_c \tau). \end{aligned}$$

- $X(t)$ is also a WSS random process.
- The power spectral density of $X(t)$ is

$$\begin{aligned} S_X(f) &= \mathcal{F}\left\{\frac{1}{2}R_A(\tau)\cos(2\pi f_c\tau)\right\} \\ &= \frac{1}{4}S_A(f+f_c)+\frac{1}{4}S_A(f-f_c). \end{aligned}$$


$S_X(f)$
$-f_c$
0
$f_c$
$f$

- Demodulation is performed as




$$Y(t) = X(t) 2\cos(2\pi f_c t).$$

We have

$$S_Y(f) = \frac{1}{2} S_X(f + f_c) + \frac{1}{2} S_X(f - f_c)$$

$$= \frac{1}{2}\left\{S_A(f+2f_c)+S_A(f)\right\}+\frac{1}{2}\left\{S_A(f)+S_A(f-2f_c)\right\}.$$

- The ideal lowpass filter passes $S_A(f)$ and blocks $S_A(f \pm 2f_c)$.
- The output of the lowpass filter has power spectral density
$$S_Y(f) = S_A(f).$$
- It can be shown that $Y(t) = X(t)$.

## Quadrature Amplitude Modulation (QAM)

- QAM signal is given by

$$X(t) = A(t)\cos(2\pi f_c t + \Theta) + B(t)\sin(2\pi f_c t + \Theta),$$

  where $A(t)$ and $B(t)$ are real-valued.

- We require that

$$\begin{aligned} R_A(\tau) &= R_B(\tau); \\ R_{B,A}(\tau) &= -R_{A,B}(\tau). \end{aligned}$$

- $S_A(f) = S_B(f)$ is a real-valued, even function of $f$
- It cab be shown that $S_{B,A}(f)$ is a purely imaginary odd function of $f$.

- The autocorrelation function is given by

$$R_X(\tau) = R_A(\tau)\cos(2\pi f_c \tau) + R_{B,A}(\tau)\sin(2\pi f_c \tau).$$

- The power spectral density is

$$S_X(f) = \frac{1}{2} S_A(f - f_c) + S_A(f + f_c) + \frac{1}{2j}\{S_{BA}(f - f_c) - S_{BA}(f + f_c)\}.$$

- Bandpass random signals arise in communication systems when wide-sense stationary white noise is filtered by bandpass filters.
- Let $N(t)$ be such process with power spectral density $S_N(f)$. Then we have

$$N(t) = N_c(t)\cos(2\pi f_c t + \Theta) - N_s(t)\sin(2\pi f_c t + \Theta),$$

where $N_c(t)$ and $N_s(t)$ are jointly wide-sense stationary processes with

$$S_{N_c}(f) = S_{N_s}(f) = \{S_N(f - f_c) + S_N(f + f_c)\}_L$$

and

$$S_{N_c,N_s}(f) = j\{S_N(f - f_c) - S_N(f + f_c)\}_L,$$

where the subscript $L$ denotes the lowpass portion of the expression in brackets.

**Example:** The received signal in an AM system is

$$Y(t) = A(t)\cos(2\pi f_c t + \Theta) + N(t),$$

where $N(t)$ is a bandlimited white noise process with spectral density

$$S_N(f) = \begin{cases} \frac{N_0}{2} & |f \pm f_c| < W \\ 0 & \text{elsewhere.} \end{cases}$$

Find the signal to noise ration of the received signal.

**Sol:** We can represent the received signal by

$$Y(t) = \{A(t) + N_c(t)\}\cos(2\pi f_c t + \Theta) - N_s(t)\sin(2\pi f_c t + \Theta).$$

The AM demodulator is used to recover $A(t)$. After multiplication by $2\cos(2\pi f_c t + \Theta)$, we have

$$\begin{aligned} 2Y(t)\cos(2\pi f_c t + \Theta) &= \{A(t) + N_c(t)\}\{1 + \cos(4\pi f_c t + 2\Theta)\} \\ &\quad - N_s(t)\sin(4\pi f_c t + 2\Theta). \end{aligned}$$

After lowpass filtering, the recovered signal is $A(t) + N_c(t)$. The power in the signal and noise components, respectively, are

$$\sigma_A^2 = \int_{-W}^{W} S_A(f)\, df$$

$$\sigma_{N_c}^2 = \int_{-W}^{W} S_{N_c}(f)\, df = \int_{-W}^{W} \left( \frac{N_0}{2} + \frac{N_0}{2} \right) df$$
$$= 2WN_0.$$

The output signal-to-noise ratio is then

$$\text{SNR} = \frac{\sigma_A^2}{2WN_0}.$$

# 7.4 Optimal Linear Systems

- By observing $\{X_{t-a}, \ldots, X_t, \ldots, X_{t+b}\}$ to obtain an estimate $Y_t$ of the desire process $Z_t$.
- The estimate $Y_t$ is required to be linear:

$$Y_t = \sum_{\beta=t-a}^{t+b} h_{t-\beta} X_\beta = \sum_{\beta=-b}^{a} h_\beta X_{t-\beta}.$$

- Mean square error is given by

$$E[e_t^2] = E[(Z_t - Y_t)^2].$$

- We seek to find the *optimal filter*, which is characterized by the impulse response $h_\beta$ that minimizes the mean square error.

**Example**: Assume that the desired signal is corrupted by noise:

$$X_\alpha = Z_\alpha + N_\alpha.$$

We are interested in estimating $Z_t$. The observation interval is $I$.

1. If $I = (-\infty, t)$ or $I = (t - a, t)$, we have a **filtering** problem.
2. If $I = (-\infty, \infty)$, we have a **smoothing** problem.
3. If $I = (t - a, t - 1)$, we have a **prediction** problem.

## The Orthogonality Condition

- Optimal filter must satisfy the **orthogonality condition**:

$$E[e_t X_\alpha] = E[(Z_t - Y_t)X_\alpha] = 0 \quad \text{for all } \alpha \in I$$

or

$$E[Z_t X_\alpha] = E[Y_t X_\alpha] \quad \text{for all } \alpha \in I.$$

- We can find that

$$E[Z_t X_\alpha] = E\left[\sum_{\beta=-b}^{a} h_\beta X_{t-\beta} X_\alpha\right] \quad \text{for all } \alpha \in I$$

$$= \sum_{\beta=-b}^{a} h_\beta E[X_{t-\beta} X_\alpha]$$

$$= \sum_{\beta=-b}^{a} h_\beta R_X(t-\alpha-\beta) \quad \text{for all } \alpha \in I.$$

- $X_\alpha$ and $Z_t$ are jointly wide-sense stationary. Therefore, we have

$$R_{Z,X}(t-\alpha) = \sum_{\beta=-b}^{a} h_\beta R_X(t-\beta-\alpha).$$

- Letting $m = t - \alpha$, we obtain the following key equation

$$R_{Z,X}(m) = \sum_{\beta=-b}^{a} h_\beta R_X(m-\beta) \qquad -b \le m \le a.$$

We have $a + b + 1$ linear equations.

## Continuous-time Estimation

- Use $Y(t)$ to estimate the desired signal $Z(t)$:

$$Y(t) = \int_{t-a}^{t+b} h(t-\beta)X(\beta)d\beta = \int_{-b}^{a} h(\beta)X(t-\beta)d\beta.$$

- It can be shown that the filter $h(\beta)$ that minimizes the mean square error is specified by

$$R_{Z,X}(\tau) = \int_{-b}^{a} h(\beta)R_X(\tau-\beta)d\beta \qquad -b \leq \tau \leq a.$$

The equation can be solved numerically.

- Determine the mean square error of the optimum filter as follows. The error $e_t$ and estimate $Y_t$ are orthogonal:

$$E[e_t Y_t] = E\left[e_t \sum h_{t-\beta} X_\beta\right] = \sum h_{t-\beta} E[e_t X_\beta] = 0.$$

- The mean square error is then

$$E[e_t^2] = E[e_t(Z_t - Y_t)] = E[e_t Z_t];$$

$$\begin{aligned} E[e_t^2] &= E[(Z_t - Y_t)Z_t] = E[Z_t Z_t] - E[Y_t Z_t] \\ &= R_Z(0) - E[Z_t Y_t] \\ &= R_Z(0) - E[Z_t \sum_{\beta=-b}^{a} h_\beta X_{t-\beta}] \end{aligned}$$

$$= \quad R_Z(0) - \sum_{\beta=-b}^{a} h_\beta R_{Z,X}(\beta).$$

- For continuous case we have

$$E[e^2(t)] = R_Z(0) - \int_{-b}^{a} h(\beta) R_{Z,X}(\beta) d\beta.$$

**Theorem**: Let $X_t$ and $Z_t$ be discrete-time, zero-mean, jointly wide-sense stationary processes, and let $Y_t$ be an estimate for $Z_t$ of the form

$$Y_t = \sum_{\beta=-b}^{a} h_\beta X_{t-\beta}.$$

The filter that minimize $E[(Z_t - Y_t)^2]$ satisfies the equation

$$R_{Z,X}(m) = \sum_{\beta=-b}^{a} h_\beta R_X(m-\beta) \qquad -b \le m \le a$$

and has mean square error given by

$$E[(Z_t - Y_t)^2] = R_Z(0) - \sum_{\beta=-b}^{a} h_\beta R_{Z,X}(\beta).$$

**Example**: Observing

$$X_\alpha = Z_\alpha + N_\alpha \qquad \alpha \in I = \{n-p, \ldots, n-1, n\}.$$

Find the set of linear equations for the optimal filter if $Z_\alpha$ and $N_\alpha$ are independent linear processes.

**Sol:** We have

$$R_{Z,X}(m) = \sum_{\beta=0}^{p} h_\beta R_X(m-\beta) \qquad m \in \{0, 1, \ldots, p\}.$$

The cross-correlation terms are

$$R_{Z,X}(m) = E[Z_n X_{n-m}] = E[Z_n(Z_{n-m} + N_{n-m})] = R_Z(m).$$

The autocorrelation terms are given by

$$R_X(m-\beta) \quad = \quad E[X_{n-\beta} X_{n-m}] = E[(Z_{n-\beta} + N_{n-\beta})(Z_{n-m} + N_{n-m})]$$

$$
\begin{aligned}
&= R_Z(m-\beta) + R_{Z,N}(m-\beta) \\
&\quad + R_{N,Z}(m-\beta) + R_N(m-\beta) \\
&= R_Z(m-\beta) + R_N(m-\beta).
\end{aligned}
$$

The $p+1$ linear equations are then

$$R_Z(m) = \sum_{\beta=0}^{p} h_\beta \{R_Z(m-\beta) + R_N(m-\beta)\} \quad m \in \{0, 1, \ldots, p\}.$$

**Example**: Let $Z_\alpha$ be a first-order autoregressive process with average power $\sigma_Z^2$ and parameter $r$ with $|r| < 1$ and $N_\alpha$ is a white noise with average power $\sigma_N^2$. Find the set of equations for the optimal filter.

**Sol:** The autocorrelation for a first-order autoregressive process is given by

$$R_Z(m) = \sigma_Z^2 r^{|m|} \qquad m = 0, \pm 1, \pm 2, \ldots.$$

The autocorrelation for the white noise is

$$R_N(m) = \sigma_N^2 \delta(m).$$

We have the $p+1$ linear equations as

$$\sigma_Z^2 r^{|m|} = \sum_{\beta=0}^{p} h_\beta \{\sigma_Z^2 r^{|m-\beta|} + \sigma_N^2 \delta(m-\beta)\} \qquad m \in \{0, \ldots, p\}.$$

Divide both sides by $\sigma_Z^2$ and Let $\Gamma = \sigma_N^2/\sigma_Z^2$, we have

$$\begin{bmatrix} 1+\Gamma & r & r^2 & \cdots & r^p \\ r & 1+\Gamma & r & \cdots & r^{p-1} \\ r^2 & r & 1+\Gamma & \cdots & r^{p-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ r^p & r^{p-1} & r^{p-2} & \cdots & 1+\Gamma \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_p \end{bmatrix} = \begin{bmatrix} 1 \\ r \\ r^2 \\ \vdots \\ r^p \end{bmatrix}.$$

## Prediction

- We want to predict $Z_n$ in terms of $Z_{n-1}, Z_{n-2}, \ldots, Z_{n-p}$:

$$Y_n = \sum_{\beta=1}^{p} h_\beta Z_{n-\beta}.$$

- For this problem $X_\alpha = Z_\alpha$ so we have

$$R_Z(m) = \sum_{\beta=1}^{p} h_\beta R_Z(m-\beta) \quad m \in \{1, \ldots, p\}.$$

In matrix form (**Yule-Walker equations**) the equations become

$$\begin{bmatrix} R_Z(1) \\ R_Z(2) \\ \vdots \\ R_Z(p) \end{bmatrix} = \begin{bmatrix} R_Z(0) & R_Z(1) & \cdots & R_Z(p-1) \\ R_Z(1) & R_Z(0) & \cdots & R_Z(p-2) \\ \vdots & \vdots & \vdots & \vdots \\ R_Z(p-1) & R_Z(p-2) & \cdots & R_Z(0) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_p \end{bmatrix}$$

$$= \boldsymbol{R} \ \boldsymbol{h}.$$

- The mean square error becomes

$$E[e_n^2] = R_Z(0) - \sum_{\beta=1}^{p} h_\beta R_Z(\beta).$$

- We can solve $\boldsymbol{h}$ by inverting the $p \times p$ matrix $\boldsymbol{R_Z}$.
- It can also be solved by **Levinson algorithm**.

## Estimation Using the Entire Realization of the Observed Process

- We want to estimate $Z_t$ by $Y_t$:

$$Y_t = \sum_{\beta=-\infty}^{\infty} h_\beta X_{t-\beta}.$$

- For continuous-time random process, we have

$$Y(t) = \int_{-\infty}^{+\infty} h(\beta) X(t-\beta) d\beta.$$

- The optimum filters are then

$$R_{Z,X}(m) = \sum_{\beta=-\infty}^{\infty} h_\beta R_X(m-\beta) \quad \text{for all } m;$$

$$R_{Z,X}(\tau) = \int_{-\infty}^{+\infty} h(\beta) R_X(\tau - \beta) d\beta \quad \text{for all } \tau.$$

- Taking Fourier transform of both sides we get

$$S_{Z,X}(f) = H(f) S_X(f).$$

- The transfer function of the optimal filter is then

$$H(f) = \frac{S_{Z,X}(f)}{S_X(f)};$$

$$h(t) = \mathcal{F}^{-1}\{H(f)\}.$$

- $h(t)$ may be noncausal.