Price and Quantity Competition in a Duopoly Homogeneous Product Market

Appendix

Proof of Lemma 1: We first derive firm 1's best-reply function.

Case 1a: Suppose $p_2 > \frac{a+c}{2}$. If firm 1 chooses $q_1 > 0$ with $p_1 < p_2$, then $q_2 = 0, q_1 = Q = a - p_1 > 0$ and $\pi_1 = (a - q_1)q_1$. Thus, firm 1's optimal output is $q_1^* = \frac{a-c}{2}$ with $p_1^* = \frac{a+c}{2} < p_2$ and equilibrium profit $\pi_1^* = \frac{(a-c)^2}{4} > 0$. If firm 1 chooses $q_1 > 0$ with $p_1 = p_2$, we have $q_1 = \frac{Q}{2} = \frac{a-p_1}{2}$ and $\pi_1 = (a - 2q_1)q_1$. Then, firm 1's optimal output is $q_1^* = \frac{(a-c)}{4}$ with $p_1^* = \frac{(a+c)}{2}$, which contradicts $p_1^* = p_2 > \frac{(a+c)}{2}$. If firm 1 chooses $q_1 = 0$, then $\pi_1 = 0$. Thus, firm 1 will choose $q_1^* = \frac{a-c}{2}$ as $p_2 > \frac{a+c}{2}$.

Case 1b: Suppose $p_2 = \frac{a+c}{2}$. Firm 1 gets $\pi_1 = 0$ by choosing $q_1 = 0$ with $p_1 > p_2$, and gets $\pi_1^* = \frac{(a-c)^2}{4}$ by choosing $q_1^* = \frac{a-c}{4}$ with $p_1 = p_2$ as in Case 1a. However, if firm 1 chooses $q_1 > 0$ with $p_1 < p_2$, then $q_1 = Q = a - p_1$. Thus, firm 1 will choose p_1 to maximize $\pi_1 = (p_1 - c)(a - p_1)$ subject to $p_1 < p_2 = \frac{a+c}{2}$. Since $\frac{\partial \pi_1}{\partial p_1} = a - 2p_1 + c > 0$ by $p_1 < \frac{a+c}{2}$, firm 1 will choose the largest p_1. However, no optimal p_1 exists due to the non-compact interval of $[0, \frac{a+c}{2})$. Therefore, firm 1 will choose $q_1^* = \frac{a-c}{4}$ as $p_2 = \frac{a+c}{2}$.

Case 1c: Suppose $p_2 \in [0, \frac{a+c}{2})$. Firm 1 will get $\pi_1 = 0$ by choosing $q_1 = 0$ with $p_1 > p_2$. If it chooses $q_1 > 0$ with $p_1 = p_2$, then $q_1^* = \frac{a-c}{4}$ and $p_1^* = p_2 = \frac{a+c}{2}$ as in Case 1a, which contradicts $p_2 < \frac{a+c}{2}$. If firm 1 chooses $q_1 > 0$ with $p_1 < p_2$, then no solution exists as in Case 1b. Thus, firm 1 will choose $q_1^* = 0$ as $p_2 \in [0, \frac{a+c}{2})$.

Cases 1a-1c imply the $R_1(p_2)$ in Lemma 1. Next, we derive firm 2's best-reply correspondence.

Case 2a: Suppose $q_1 = 0$. Firm 2 is a monopolist with $\pi_2 = (a - p_2)(p_2 - c)$. Thus, firm 2's optimal price is $p_2^* = \frac{a+c}{2}$ with $q_2^* = \frac{a-c}{2}$ and $\pi_2^* = \frac{(a-c)^2}{4}$.

Case 2b: Suppose $q_1 \in (0, \frac{a-c}{4})$. If firm 2 chooses $p_2 > p_1$, then $q_2 = 0$ and $\pi_2 = 0$. If firm 2 chooses $p_2 = p_1$, then $q_2 = \frac{Q}{2}$ and $\pi_2 = \frac{(p_2-c)(a-p_2)}{2}$. Thus, firm 2's optimal price is $p_2^* = \frac{a+c}{2}$ with $q_2^* = \frac{a-c}{4} = q_1$, which contradicts $q_1 < \frac{a-c}{4}$. If firm 2 chooses $p_2 < p_1$, then $q_2 = Q$ and $q_1 = 0$, which contradicts $q_1 > 0$. Therefore, firm 2 will choose $p_2 > p_1 \in \left(\frac{3a+c}{4}, a\right)$ as $q_1 \in \left(0, \frac{a-c}{4}\right)$.

Case 2c: Suppose $q_1 = \frac{a-c}{4}$. If firm 2 chooses $p_2 > p_1$, then $q_2 = 0$ and $\pi_2 = 0$. If firm 2 chooses $p_2 = p_1$, then $p_2^* = \frac{a+c}{2}$ with $q_2^* = \frac{a-c}{4} = q_1$ and $\pi_2^* = \frac{(a-c)^2}{8} > 0$ as in Case 2b. No solution exists if firm 2 chooses $p_2 < p_1$ as in Case 2b. Thus, firm 2 will choose $p_2 = \frac{a+c}{2}$ as $q_1 = \frac{a-c}{4}$.

Case 2d: Suppose $q_1 \in \left(\frac{a-c}{4}, a-c\right)$. If firm 2 chooses $p_2 > p_1$, then $q_2 = 0$ and $\pi_2 = 0$. If firm 2 chooses $p_2 = p_1$, then its optimal price is $p_2^* = \frac{a+c}{2}$ with $q_2^* = \frac{a-c}{4} = q_1$, which contradicts $q_1 > \frac{a-c}{4}$. As in Case 2b, no solution exists if firm 2 chooses $p_2 < p_1$. Therefore, firm 2 will choose $p_2 > p_1 \in \left(c, \frac{3a+c}{4}\right)$ as $q_1 \in \left(\frac{a-c}{4}, a-c\right)$.

Case 2e: Suppose $q_1 \in [a-c, a]$. If firm 2 chooses $p_2 > p_1$, then $q_2 = 0$ and $\pi_2 = 0$. As in Case 2d, no solution exists if firm 2 chooses $p_2 = p_1$ or $p_2 < p_1$. Therefore, firm 2 will choose $p_2 > p_1 \in [0, c]$ as $q_1 \in [a-c, a]$.

Cases 2a-2e imply the $R_2(q_1)$ in Lemma 1. The intersections of two firms' best-reply correspondence/function give the Cournot-Bertrand equilibria stated in Lemma 1.

Proof of Proposition 1: At $\left(q_1^{CB} = \frac{a-c}{4}, p_2^{CB} = \frac{a+c}{2}\right)$, we have $p^{CB} = \frac{a+c}{2} > p^C = \frac{a+2c}{3} > p^B = c$, $Q^B = (a-c) > Q^C = \frac{2(a-c)}{3} > Q^{CB} = \frac{(a-c)}{2}$, $CS^B = \frac{(a-c)^2}{2} > CS^C = \frac{2(a-c)^2}{9} > CS^{CB} = \frac{(a-c)^2}{8}$ and $SW^B = \frac{(a-c)^2}{2} > SW^C = \frac{4(a-c)^2}{9} > SW^{CB} = \frac{3(a-c)^2}{8}$. In contrast, at $\left(q_1^{CB} = \frac{a-c}{2}, p_2^{CB} > \frac{a+c}{2}\right)$, we have $p^{CB} = \frac{a+c}{2} > p^C = \frac{a+2c}{3} > p^B = c$, $Q^B = (a-c) > Q^C = \frac{2(a-c)}{3} > Q^{CB} = \frac{(a-c)}{2}$, $CS^B = \frac{(a-c)^2}{2} > CS^C = \frac{2(a-c)^2}{9} > CS^{CB} = \frac{(a-c)^2}{8}$ and $SW^B = \frac{(a-c)^2}{2} > SW^C = \frac{4(a-c)^2}{9} > SW^{CB} = \frac{3(a-c)^2}{8}$.

Proof of Proposition 2: The first part is because of $\frac{(a-c)^2}{9} > 0$, and the second part is due to $\frac{(a-c)^2}{9} < \frac{(a-c)^2}{8}$ and $\frac{(a-c)^2}{8} > 0$.

Proof of Lemma 2: We first derive the Cournot-Bertrand equilibria under the efficient tie-breaking rule.

Lemma A. *Suppose that Cournot firm 1 and Bertrand firm 2 produce a homogeneous product with respective marginal costs* c_1 *and* c_2, *and* $a > c_1 > c_2 > 0$. *The efficient tie-breaking rule is adopted. Then firm 1's best-reply function* $R_1(p_2)$ *is*

$$q_1 = R_1(p_2) = \begin{cases} \frac{a-c_1}{2} & if \;\; p_2 > \frac{a+c_1}{2}, \\ 0 & if \;\; p_2 \le \frac{a+c_1}{2}, \end{cases}$$

and firm 2's best-reply correspondence $R_2(q_1)$ *is*

$$p_2 = R_2(q_1) \begin{cases} = \frac{a+c_2}{2} & if \;\; q_1 = 0, \\ > p_1 \in \left(\frac{a+c_1}{2},\, a\right) & if \;\; q_1 \in \left(0,\, \frac{a-c_1}{2}\right), \\ > p_1 = \frac{a+c_1}{2} & if \;\; q_1 = \frac{a-c_1}{2}, \\ > p_1 \in \left(c_1,\, \frac{a+c_1}{2}\right) & if \;\; q_1 \in \left(\frac{a-c_1}{2},\, a-c_1\right), \\ > p_1 \in [0,\, c_1] & if \;\; q_1 \in [a-c_1,\, a]. \end{cases}$$

Accordingly, the first Cournot-Bertrand equilibrium is $\left(q_C^{CB} = \frac{a-c_1}{2},\, p_B^{CB} > \frac{a+c_1}{2}\right)$ *with* $\left(p_C^{CB} = \frac{a+c_1}{2},\, q_B^{CB} = 0\right)$ *and* $\left(\pi_C^{CB} = \frac{(a-c_1)^2}{4},\, \pi_B^{CB} = 0\right)$*, and the second equilibrium is* $\left(q_C^{CB} = 0,\, p_B^{CB} = \frac{a+c_2}{2}\right)$ *with* $\left(p_C^{CB} = \frac{a+c_2}{2},\, q_B^{CB} = \frac{a-c_2}{2}\right)$ *and* $\left(\pi_C^{CB} = 0,\, \pi_B^{CB} = \frac{(a-c_2)^2}{4}\right)$.

Proof. Firm 1's best-reply function is first derived below.

Case 1a: Suppose $p_2 > \frac{a+c_1}{2}$. If firm 1 chooses $q_1 = 0$ with $p_1 > p_2$, $p_1 = p_2 \ge a$ or $p_1 = p_2 < a$, then $\pi_1 = 0$. If firm 1 chooses $q_1 > 0$ with $p_1 < \min\{a, p_2\}$, then p_1 will be selected to maximize $\pi_1 = (p_1 - c_1)(a - p_1)$ subject to $p_1 < p_2$. The optimal solution is $q_1^* = \frac{(a-c_1)}{2}$ with $p_1^* = \frac{(a+c_1)}{2}$ and $\pi_1^* = \frac{(a-c_1)^2}{4} > 0$. Thus, firm 1 will choose $q_1^* = \frac{a-c_1}{2}$ as $p_2 > \frac{a+c_1}{2}$.

Case 1b: Suppose $p_2 = \frac{a+c_1}{2}$. Firm 1 will get $\pi_1 = 0$ by choosing $q_1 = 0$ with $p_1 \ge p_2$. For $q_1 > 0$, firm 1 will choose p_1 to maximize $\pi_1 = (p_1 - c_1)(a - p_1)$ subject to $p_1 < p_2 = \frac{a+c_1}{2}$. Since interval $[0, \frac{(a+c_1)}{2})$ is not compact, no solution exists. That is, firm 1 will choose $q_1^* = 0$ as $p_2 = \frac{a+c_1}{2}$.

Case 1c: Suppose $p_2 \in [0, \frac{a+c_1}{2})$. As in Case 1b, there is no solution if firm 1 chooses $q_1 > 0$. Thus, firm 1 will choose $q_1^* = 0$ as $p_2 < \frac{a+c_1}{2}$.

The results of Cases 1a-1c imply the $R_1(p_2)$ in Lemma A.

Next, firm 2's best-reply correspondence is derived as follows.

Case 2a: Suppose $q_1 = 0$. Firm 2 is a monopolist with $\pi_2 = (a - p_2)(p_2 - c_2)$. Thus, its optimal price is $p_2^* = \frac{a+c_2}{2}$ with $q_2^* = \frac{a-c_2}{2}$ and $\pi_2^* = \frac{(a-c_2)^2}{4}$.

Case 2b: Suppose $q_1 \in \left(0, \frac{a-c_1}{2}\right)$. If firm 2 chooses $p_2 > p_1$, then $q_2 = 0$ and $\pi_2 = 0$. If firm 2 chooses $p_2 \le p_1$, then $q_2 = Q$ and $q_1 = 0$, which contradicts $q_1 > 0$. Thus, it will choose $p_2 > p_1 \in \left(\frac{a+c_1}{2}, a\right)$ with $q_2 = 0$ as $q_1 \in \left(0, \frac{a-c_1}{4}\right)$.

Case 2c: Suppose $q_1 = \frac{a-c_1}{2}$. As in Case 2b, firm 2 will choose $p_2 > p_1 = \frac{a+c_1}{2}$ with $q_2 = 0$.

Case 2d: Suppose $q_1 \in \left(\frac{a-c_1}{2}, a-c_1\right)$. As in Case 2b, firm 2 will choose $p_2 > p_1 \in \left(c_1, \frac{a+c_1}{2}\right)$ with $q_2 = 0$.

Case 2e: Suppose $q_1 \in [a-c_1, a]$. As in Case 2b, firm 2 will choose $p_2 > p_1 \in [0, c_1]$ with $q_2 = 0$.

The results of Cases 2a-2e imply the $R_2(q_1)$ in Lemma A. Firms' best-reply function/correspondence and two Cournot-Bertrand equilibria are drawn in Figure 2.

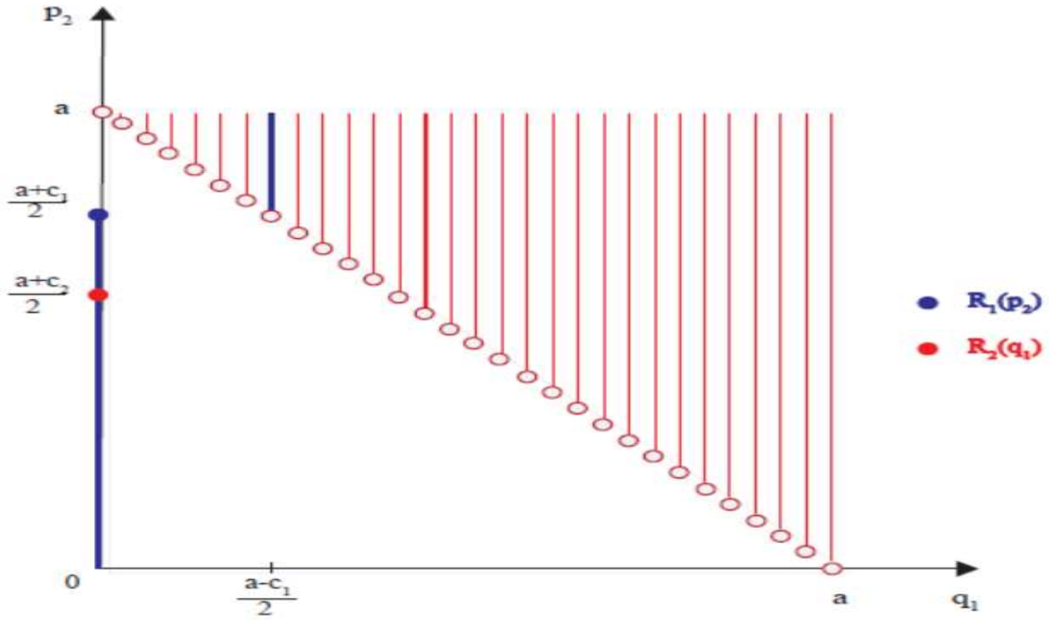

Figure 2

Second, we derive the Cournot-Bertrand equilibria under the equal-sharing tie-breaking rule.

Lemma B. *Suppose that Cournot firm 1 and Bertrand firm 2 produce a homogeneous product with respective marginal costs* c_1 *and* c_2, *and* $a > c_1 > c_2 > 0$. *The equal-sharing tie-breaking rule is adopted. Then firm 1's best-reply function* $R_1(p_2)$ *is*

$$q_1 = R_1(p_2) = \begin{cases} \frac{a-c_1}{2} & if \;\; p_2 > \frac{a+c_1}{2}, \\ \frac{a-c_1}{4} & if \;\; p_2 = \frac{a+c_1}{2}, \\ 0 & if \;\; p_2 < \frac{a+c_1}{2}, \end{cases}$$

and firm 2's best-reply correspondence $R_2(q_1)$ *is*

$$p_2 = R_2(q_1)\begin{cases} = \frac{a+c_2}{2} & if\ \ q_1 = 0, \\ = p_1 \in (c_1, a) & if\ \ q_1 \in \left(0, \frac{a-c_1}{2}\right), \\ = p_1 = c_1 & if\ \ q_1 = \frac{a-c_1}{2}, \\ = p_1 \in (c_2, c_1) & if\ \ q_1 \in \left(\frac{a-c_1}{2}, \frac{a-c_2}{2}\right), \\ = c_2\ or > \frac{a+c_2}{2} & if\ \ q_1 = \frac{a-c_2}{2}, \\ > p_1 \in [0, \frac{a+c_2}{2}) & if\ \ q_1 \in (\frac{a-c_2}{2}, a]. \end{cases}$$

Accordingly, the first Cournot-Bertrand equilibrium is $\left(q_C^{CB} = \frac{a-c_1}{4}, p_B^{CB} = \frac{a+c_1}{2}\right)$ *with* $\left(p_C^{CB} = \frac{a+c_1}{2}, q_B^{CB} = \frac{a-c_1}{4}\right)$ *and* $\left(\pi_C^{CB} = \frac{(a-c_1)^2}{8}, \pi_B^{CB} = \frac{(a-c_1)^2}{8}\right)$*, and the second equilibrium is* $\left(q_C^{CB} = 0,\ p_B^{CB} = \frac{a+c_2}{2}\right)$ *with* $\left(p_C^{CB} = \frac{a+c_2}{2}, q_B^{CB} = \frac{a-c_2}{2}\right)$ *and* $\left(\pi_C^{CB} = 0,\ \pi_B^{CB} = \frac{(a-c_2)^2}{4}\right)$.

Proof. Firm 1's best-reply function is derived below.

<u>Case 1a</u>: Suppose $p_2 > \frac{a+c_1}{2}$. As in Case 1a of Lemma A, firm 1 will choose $q_1^* = \frac{(a-c_1)}{2}$ with $p_1^* = \frac{(a+c_1)}{2}$ and $\pi_1^* = \frac{(a-c_1)^2}{4} > 0$.

<u>Case 1b</u>: Suppose $p_2 = \frac{a+c_1}{2}$. Firm 1 will get $\pi_1 = 0$ by choosing $q_1 = 0$ with $p_1 > p_2$. If firm 1 chooses $q_1 > 0$ with $p_1 = p_2 < a$. Then, p_1 will be selected to maximize $\pi_1 = \frac{(p_1-c_1)(a-p_1)}{2}$. The optimal solution is $q_1^* = \frac{(a-c_1)}{4}$ with $p_1^* = p_2 = \frac{(a+c_1)}{2}$ and $\pi_1^* = \frac{(a-c_1)^2}{8} > 0$. If firm 1 chooses $q_1 > 0$ with $p_1 < p_2$, then no solution exists as in Case 1b of Lemma A. Thus, firm 1 will choose $q_1^* = \frac{(a-c_1)}{4}$ as $p_2 = \frac{a+c_1}{2}$.

<u>Case 1c</u>: Suppose $p_2 \in [0, \frac{a+c_1}{2})$. If firm 1 chooses $q_1 > 0$ with $p_1 = p_2$, then $p_1^* = p_2 = \frac{(a+c_1)}{2}$ as in Case 1b, which contradicts $p_2 < \frac{a+c_1}{2}$. In contrast, if firm 1 chooses $q_1 > 0$ with $p_1 < p_2$, then no solution exists as in Case 1b. Thus, it is optimal for firm 1 to choose $q_1^* = 0$.

The results of Cases 1a-1c imply the $R_1(p_2)$ in Lemma B.

Next, firm 2's best-reply correspondence is derived as follows.

<u>Case 2a</u>: Suppose $q_1 = 0$. As in Case 2a of Lemma A, firm 2's optimal price is $p_2^* = \frac{a+c_2}{2}$ with $q_2^* = \frac{a-c_2}{2}$ and $\pi_2^* = \frac{(a-c_2)^2}{4}$.

<u>Case 2b</u>: Suppose $q_1 \in \left(0, \frac{a-c_1}{2}\right)$. If firm 2 chooses $p_2 > p_1$, then $q_2 = 0$ and $\pi_2 = 0$. If firm 2 chooses $p_2 = p_1$, then $q_1 = q_2 = \frac{Q}{2}$ and $\pi_2 = (a - 2q_2)q_2 > 0$ with $p_1 = a - 2q_1 \in (c_1, a)$ by $q_1 \in \left(0, \frac{a-c_1}{2}\right)$. If firm 2 chooses $p_2 < p_1$, then $q_2 = Q$ and $q_1 = 0$, which contradicts $q_1 > 0$. Thus,

firm 2 will choose $p_2 = p_1 \in (c_1, a)$ as $q_1 \in \left(0, \frac{a-c_1}{2}\right)$.

Case 2c: Suppose $q_1 = \frac{a-c_1}{2}$. As in Case 2b, firm 2 will choose $p_2 = p_1 = c_1$.

Case 2d: Suppose $q_1 \in \left(\frac{a-c_1}{2}, \frac{a-c_2}{2}\right)$. As in Case 2b, firm 2 will choose $p_2 = p_1 \in (c_2, c_1)$.

Case 2e: Suppose $q_1 = \frac{a-c_2}{2}$. Firm 2 will choose $p_2 \geq p_1 = c_2$ with $\pi_2 = 0$.

Case 2f: Suppose $q_1 \in (\frac{a-c_2}{2}, a]$. If firm 2 chooses $p_2 > p_1$, then $q_2 = 0$ and $\pi_2 = 0$. If firm 2 chooses $p_2 = p_1$, then $q_1 = q_2 = \frac{Q}{2}$ and $p_2 = p_1 = a - 2q_1 \in (-a, c_2)$ by $q_1 \in (\frac{a-c_2}{2}, a]$, which suggests $\pi_2 < 0$. If firm 2 chooses $p_2 < p_1$, then $q_2 = Q$ and $q_1 = 0$, which contradicts $q_1 > 0$. Thus, firm 2 will choose $p_2 > p_1 \in [0, \frac{a+c_2}{2})$ as $q_1 \in (\frac{a-c_2}{2}, a]$.

The results of Cases 2a-2f imply the $R_2(q_1)$ in Lemma B. Firms' best-reply function/correspondence and two Cournot-Bertrand equilibria are drawn in Figure 3.

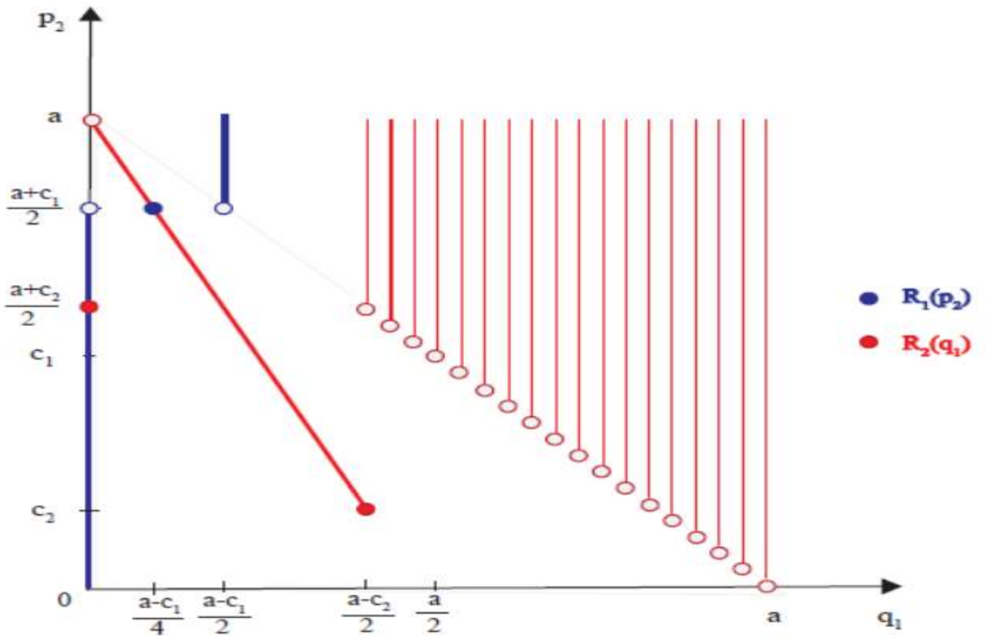

Figure 3

Accordingly, Lemmas A and B suggest $\left(q_C^{CB} = 0,\ p_B^{CB} = \frac{a+c_2}{2}\right)$ *with* $\left(p_C^{CB} = \frac{a+c_2}{2},\ q_B^{CB} = \frac{a-c_2}{2}\right)$ *and* $\left(\pi_C^{CB} = 0,\ \pi_B^{CB} = \frac{(a-c_2)^2}{4}\right)$ surviving under the two tie-breaking rules. These prove Lemma 2.

<u>Proof of Lemma 3</u>: We first derive the Cournot-Bertrand equilibria under the efficient tie-breaking rule.

Lemma C. *Suppose that Cournot firm 1 and Bertrand firm 2 produce a homogeneous product with* *respective marginal costs* c_1 *and* c_2*, and* $a > c_1 > c_2 > 0$. *The efficient tie-breaking rule is adopted. Then firm 1's best-reply function* $R_1(p_2)$ *is*

$$q_1 = R_1(p_2) = \begin{cases} \frac{a-c_1}{2} & \text{if } p_2 \geq \frac{a+c_1}{2}, \\ Q = a - p_2 & \text{if } c_1 \leq p_2 < \frac{a+c_1}{2}, \\ 0 & \text{if } 0 \leq p_2 < c_1, \end{cases}$$

and firm 2's best-reply correspondence $R_2(q_1)$ *is*

$$p_2 = R_2(q_1) \begin{cases} = \frac{a+c_2}{2} & \text{if } q_1 = 0, \\ \geq p_1 \in \left(\frac{a+c_1}{2}, a\right) & \text{if } q_1 \in (0, \frac{a-c_1}{2}), \\ \geq p_1 = \frac{a+c_1}{2} & \text{if } q_1 = \frac{a-c_1}{2}, \\ \geq p_1 \in \left(c_1, \frac{a+c_1}{2}\right) & \text{if } q_1 \in \left(\frac{a-c_1}{2}, a - c_1\right), \\ \geq p_1 \in [0, c_1] & \text{if } q_1 \in [a - c_1, a]. \end{cases}$$

Accordingly, the first Cournot-Bertrand equilibrium is $\left(q_C^{CB} = \frac{a-c_1}{2},\, p_B^{CB} \geq \frac{a+c_1}{2}\right)$ *with* $\left(p_C^{CB} = \frac{a+c_1}{2},\, q_B^{CB} = 0\right)$ *and* $\left(\pi_C^{CB} = \frac{(a-c_1)^2}{4},\, \pi_B^{CB} = 0\right)$*, and the second equilibrium is* $\left(q_C^{CB} = a - p_B^{CB}, p_B^{CB} \in [c_1, \frac{a+c_1}{2})\right)$ *with* $\left(p_C^{CB} = p_B^{CB},\, q_B^{CB} = 0\right)$ *and* $\left(\pi_C^{CB} = q_C^{CB}\left[p_C^{CB} - c_1\right],\, \pi_B^{CB} = 0\right)$.

Proof. Firm 1's best-reply function is first derived below.

Case 1a: Suppose $p_2 > \frac{a+c_1}{2}$. If firm 1 chooses $q_1 = 0$ with $p_1 > p_2$ or $p_1 = p_2 \geq a$, then $\pi_1 = 0$. If firm 1 chooses $q_1 > 0$ with $p_1 = p_2 < a$, then p_1 will be selected to maximize $\pi_1 = (p_1 - c_1)(a - p_1)$ subject to $p_1 = p_2$. The optimal solution is $p_1^* = \frac{(a+c_1)}{2} = p_2$ with $q_1^* = \frac{(a-c_1)}{2}$, which contradicts $p_2 > \frac{a+c_1}{2}$. If firm 1 chooses $q_1 > 0$ with $p_1 < p_2$, then p_1 will be selected to maximize $\pi_1 = (p_1 - c_1)(a - p_1)$ subject to $p_1 < p_2$. The optimal solution is $p_1^* = \frac{(a+c_1)}{2} < p_2$ with $q_1^* = \frac{(a-c_1)}{2}$ and $\pi_1^* = \frac{(a-c_1)^2}{4}$ >0. Thus, firm 1 will choose $q_1^* = \frac{a-c_1}{2}$ as $p_2 > \frac{a+c_1}{2}$.

Case 1b: Suppose $p_2 = \frac{a+c_1}{2}$. If firm 1 chooses $q_1 = 0$ with $p_1 > p_2$, then $\pi_1 = 0$. If firm 1 chooses $q_1 > 0$ with $p_1 = p_2$, then p_1 will be selected to maximize $\pi_1 = (p_1 - c_1)(a - p_1)$ subject to $p_1 = p_2$. The optimal solution is $p_1^* = \frac{(a+c_1)}{2} = p_2$ with $q_1^* = \frac{(a-c_1)}{2}$ and $\pi_1^* = \frac{(a-c_1)^2}{4}$ >0. However, if firm 1 chooses $q_1 > 0$ with $p_1 < p_2$, then p_1 will be selected to maximize $\pi_1 = (p_1 - c_1)(a - p_1)$ subject to $p_1 > p_2 = \frac{a+c_2}{2}$. Since $\frac{\partial \pi_1}{\partial p_1} > 0$ and interval $[0, p_2)$ is not compact, no solution exists. Thus, the optimal solution is $q_1^* = \frac{(a-c_1)}{2}$ as $p_2 = \frac{a+c_1}{2}$.

Case 1c: Suppose $p_2 \in [c_1, \frac{a+c_1}{2})$. As in Case 1b, firm 1 will choose $p_1^* = p_2 \in [0, \frac{a+c_1}{2})$ with $q_1^* = a - p_2$ and $\pi_1^* = (p_1 - c_1) q_1^* > 0$.

Case 1d: Suppose $p_2 \in [0, c_1)$. If firm 1 chooses $q_1 = 0$ with $p_1 > p_2$, then $\pi_1 = 0$. If firm 1 chooses $q_1 > 0$ with $p_1 = p_2$, then $-c_1 \le p_1 - c_1 < 0$ by $p_2 \in [0, c_1)$. Hence $\pi_1 < 0$. We have $\pi_1 < 0$ as well if firm 1 chooses $q_1 > 0$ with $p_1 < p_2$. Thus, it is optimal for firm 1 to choose $q_1^* = 0$ as $p_2 \in [0, c_1)$.

The results of Cases 1a-1d imply the $R_1(p_2)$ in Lemma C.

Next, firm 2's best-reply correspondence is derived as follows.

Case 2a: Suppose $q_1 = 0$. As in Case 2a of Lemma A, firm 2 will choose $p_2^* = \frac{a+c_2}{2}$ as $q_1 = 0$.

Case 2b: Suppose $q_1 \in \left(0, \frac{a-c_1}{2}\right)$. If firm 2 chooses $p_2 > p_1$, then $q_2 = 0$ and $\pi_2 = 0$. If firm 2 chooses $p_2 = p_1$, then $q_2 = 0$ and $\pi_2 = 0$. If firm 2 chooses $p_2 < p_1$, then $q_2 = Q$ and $q_1 = 0$, which contradicts $q_1 > 0$. Thus, firm 2 will choose $p_2 \ge p_1 \in \left(\frac{a+c_1}{2}, a\right)$ with $q_2 = 0$ as $q_1 \in \left(0, \frac{a-c_1}{2}\right)$.

Case 2c: Suppose $q_1 = \frac{a-c_1}{2}$. As in Case 2b, firm 2 will choose $p_2 \ge p_1 = \frac{a+c_1}{2}$ with $q_2 = 0$.

Case 2d: Suppose $q_1 \in \left(\frac{a-c_1}{2}, a - c_1\right)$. As in Case 2b, firm 2 will choose $p_2 \ge p_1 \in \left(c_1, \frac{a+c_1}{2}\right)$ with $q_2 = 0$.

Case 2e: Suppose $q_1 \in [a - c_1, a]$. As in Case 2b, firm 2 will choose $p_2 \ge p_1 \in [0, c_1]$ with $q_2 = 0$.

The results of Cases 2a-2e imply the $R_2(q_1)$ in Lemma C. Firms' best-reply function/correspondence and two Cournot-Bertrand equilibria are drawn in Figure 4.

Second, we derive the Cournot-Bertrand equilibria under the equal-sharing tie-breaking rule.

Lemma D. *Suppose that Cournot firm 1 and Bertrand firm 2 produce a homogeneous product with respective marginal costs* c_1 *and* c_2*, and* $a > c_1 > c_2 > 0$ *. The equal-sharing tie-breaking rule is adopted. Then firm 1's best-reply function* $R_1(p_2)$ *is*

$$q_1 = R_1(p_2) = \begin{cases} \frac{a-c_1}{2} & \text{if } p_2 > \frac{a+c_1}{2}, \\ \frac{a-c_1}{4} & \text{if } p_2 = \frac{a+c_1}{2}, \\ 0 & \text{if } p_2 < \frac{a+c_1}{2}, \end{cases}$$

and firm 2's best-reply correspondence $R_2(q_1)$ *is*

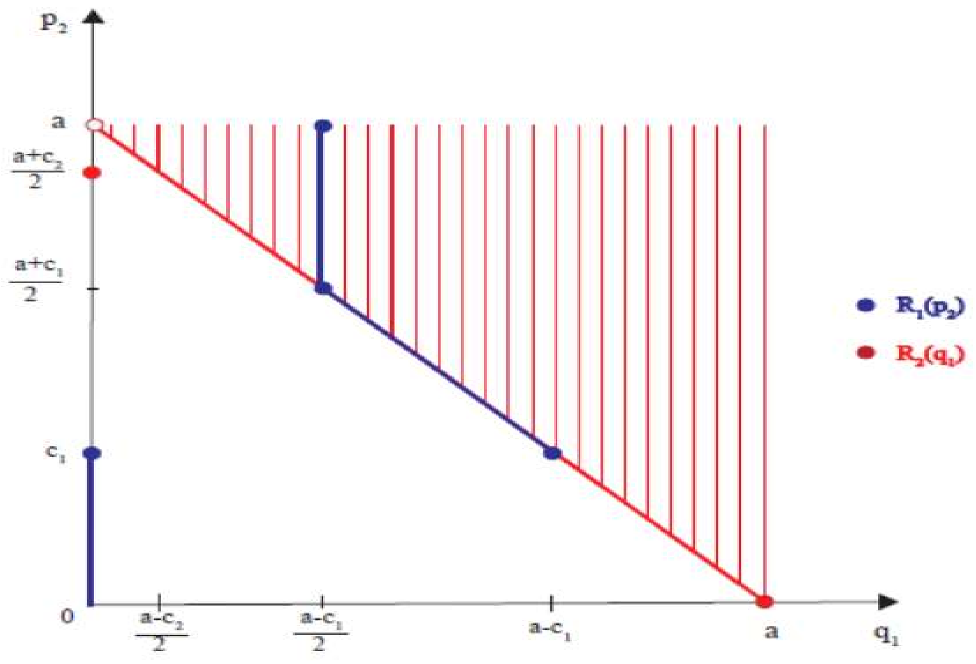

Figure 4

$$p_2 = R_2(q_1)\begin{cases} = \frac{a+c_2}{2} & if \;\; q_1 = 0, \\ = a - 2q_1 & if \;\; q_1 \in (0, \frac{a-c_2}{2}], \\ > p_1 \in \left(\frac{a+c_1}{2}, \frac{a+c_2}{2}\right) & if \;\; q_1 \in \left(\frac{a-c_2}{2}, \frac{a-c_1}{2}\right), \\ > p_1 \in \left[0, \frac{a+c_1}{2}\right] & if \;\; q_1 \in \left[\frac{a-c_1}{2}, a\right]. \end{cases}$$

Accordingly, the first Cournot-Bertrand equilibrium is $\left(q_C^{CB} = \frac{a-c_1}{2},\, p_B^{CB} > \frac{a+c_1}{2}\right)$ *with* $\left(p_C^{CB} = \frac{a+c_1}{2},\, q_B^{CB} = 0\right)$ *and* $\left(\pi_C^{CB} = \frac{(a-c_1)^2}{4},\, \pi_B^{CB} = 0\right)$*, and the second equilibrium is* $\left(q_C^{CB} = \frac{a-c_1}{4},\, p_B^{CB} = \frac{a+c_1}{2}\right)$ *with* $\left(p_C^{CB} = \frac{a+c_1}{2},\, q_B^{CB} = \frac{a-c_1}{4}\right)$ *and* $\left(\pi_C^{CB} = \frac{(a-c_1)^2}{8},\, \pi_B^{CB} = \frac{(a-c_1)^2}{8}\right)$.

Proof. Firm 1's best-reply function is first derived below.

Case 1a: Suppose $p_2 > \frac{a+c_1}{2}$. As in Case 1a of Lemma B, firm 1 will choose $q_1^* = \frac{(a-c_1)}{2}$ with $p_1^* = \frac{(a+c_1)}{2}$ and $\pi_1^* = \frac{(a-c_1)^2}{4} > 0$.

Case 1b: Suppose $p_2 = \frac{a+c_1}{2}$. As in Case 1b of Lemma B, firm 1 will choose $q_1^* = \frac{(a-c_1)}{4}$ with $p_1^* = \frac{(a+c_1)}{2}$ and $\pi_1^* = \frac{(a-c_1)^2}{8} > 0$.

Case 1c: Suppose $p_2 \in [0, \frac{a+c_1}{2})$. As in Case 1c of Lemma B, firm 1 will choose $q_1^* = 0$.

The results of Cases 1a-1c imply the $R_1(p_2)$ in Lemma D.

Next, firm 2's best-reply correspondence is derived as follows.

Case 2a: Suppose $q_1 = 0$. As in Case 2a of Lemma B, firm 2's optimal price is $p_2^* = \frac{a+c_2}{2}$ with $q_2^* = \frac{a-c_2}{2}$ and $\pi_2^* = \frac{(a-c_2)^2}{4}$.

Case 2b: Suppose $q_1 \in \left(0, \frac{a-c_2}{2}\right)$. If firm 2 chooses $p_2 > p_1$, then $q_2 = 0$ and $\pi_2 = 0$. If firm 2 chooses $p_2 = p_1$, then $q_1 = q_2 = \frac{Q}{2}$ and $\pi_2 = (a - 2q_2)q_2 > 0$ with $p_1 = a - 2q_1 \in (c_2, a)$ by $q_1 \in \left(0, \frac{a-c_2}{2}\right)$. If firm 2 chooses $p_2 < p_1$, then $q_2 = Q$ and $q_1 = 0$, which contradicts $q_1 > 0$. Thus, firm 2 will choose $p_2 = p_1 \in (c_2, a)$ as $q_1 \in \left(0, \frac{a-c_2}{2}\right)$.

Case 2c: Suppose $q_1 = \frac{a-c_2}{2}$. As in Case 2b, firm 2 will choose $p_2 = p_1 = c_2$.

Case 2d: Suppose $q_1 \in \left(\frac{a-c_2}{2}, \frac{a-c_1}{2}\right)$. If firm 2 chooses $p_2 > p_1$, then $q_2 = 0$ and $\pi_2 = 0$. If firm 2 chooses $p_2 = p_1$, then $q_1 = q_2 = \frac{Q}{2}$ and $p_2 = a - 2q_2 \in (c_1, c_2)$ by $q_1 \in \left(\frac{a-c_2}{2}, \frac{a-c_1}{2}\right)$, which suggests $\pi_2 < 0$. If firm 2 chooses $p_2 < p_1$, then $q_2 = Q$ and $q_1 = 0$, which contradicts $q_1 > 0$. Thus, firm 2 will choose $p_2 > p_1 \in \left(\frac{a+c_1}{2}, \frac{a+c_2}{2}\right)$ as $q_1 \in \left(\frac{a-c_2}{2}, \frac{a-c_1}{2}\right)$.

Case 2e: Suppose $q_1 \in \left[\frac{a-c_1}{2}, a\right]$. If firm 2 chooses $p_2 > p_1$, then $q_2 = 0$ and $\pi_2 = 0$. As in Case 2d, $\pi_2 < 0$ if firm 2 chooses $p_2 = p_1$. If firm 2 chooses $p_2 < p_1$, then $q_2 = Q$ and $q_1 = 0$, which contradicts $q_1 > 0$. Thus, firm 2 will choose $p_2 > p_1 \in \left[0, \frac{a+c_1}{2}\right]$ as $q_1 \in \left[\frac{a-c_1}{2}, a\right]$.

The results of Cases 2a-2e imply the $R_2(q_1)$ in Lemma D. Firms' best-reply function/correspondence and two Cournot-Bertrand equilibria are drawn in Figure 5.

Accordingly, Lemmas C and D imply that $\left(q_C^{CB} = \frac{a-c_1}{2},\ p_B^{CB} > \frac{a+c_1}{2}\right)$ with $\left(p_C^{CB} = \frac{a+c_1}{2},\ q_B^{CB} = 0\right)$ and $\left(\pi_C^{CB} = \frac{(a-c_1)^2}{4},\ \pi_B^{CB} = 0\right)$ surviving under the two tie-breaking rules. These prove Lemma 3.

Proof of Proposition 4: There are two cases according to relative sizes of c_1 and c_2.

Case 1: Suppose $a > c_1 > c_2 > 0$ and $a > 2c_1 - c_2$. First, since $p^C = \frac{a+c_1+c_2}{3}$, $p^B = c_1$ and $p^{CB} = \frac{a+c_2}{2}$, we have $p^C - p^B = \frac{a+c_2-2c_1}{3} > 0$, $p^B - p^{CB} = \frac{2c_1-a-c_2}{2} < 0$ and $p^C - p^{CB} = \frac{-a-c_2+2c_1}{6} < 0$ by $a > 2c_1 - c_2$. Thus, $p^{CB} > p^C > p^B$. Second, since $Q^C = \frac{2a-c_1-c_2}{3}$, $Q^B = a - c_1$ and $Q^{CB} = \frac{a-c_2}{2}$, we have $Q^C - Q^B = \frac{-a-c_2+2c_1}{3} < 0$, $Q^B - Q^{CB} = \frac{a+c_2-2c_1}{2} > 0$ and $Q^C - Q^{CB} = \frac{a+c_2-2c_1}{6} > 0$ by $a > 2c_1 - c_2$. Thus, we have $Q^B > Q^C > Q^{CB}$. Third, since

$Q^B > Q^C > Q^{CB}$, we have $CS^B > CS^C > CS^{CB}$. Fourth, some calculations show $SW^{CB} - SW^B = \frac{[a-c_2+2(c_1-c_2)][2c_1-a-c_2]}{8} < 0$ by $a > 2c_1 - c_2$ and $a > c_1 > c_2 > 0$. Moreover, $SW^B - SW^C = \frac{1}{18}\left\{-3(a-c_1)^2 + 5(a-c_1)(a-c_2) + 11(a-c_1)(c_1-c_2) - 4(a-c_2)(c_1-c_2) - 6(c_1-c_2)^2\right\}$ $> \frac{1}{18}\left\{2(a-c_1)^2 + (c_1-c_2)^2\right\} > 0$ by $a > 2c_1 - c_2$ and $a > c_1 > c_2 > 0$. And $SW^C - SW^{CB} = \frac{1}{72}\left\{12(a-c_1)^2 - 15(a-c_2)^2 + 32(c_1-c_2)^2 + 8(a-c_1)(a-c_2)\right\} =$ $\frac{1}{72}\left\{\left[2(a-c_1)+3(a-c_2)\right]\left[6(a-c_1)-5(a-c_2)\right] + 32(c_1-c_2)^2\right\} > \frac{1}{72}\left\{-20(c_1-c_2)^2 + 32(c_1-c_2)^2\right\} =$ $\frac{1}{72}\left[12(c_1-c_2)^2\right] > 0$. These imply $SW^B > SW^C > SW^{CB}$.

Case 2: Suppose $a > c_2 > c_1 > 0$ and $a > 2c_2 - c_1$. Using the same method, we can obtain the results similar to those in Case 1.

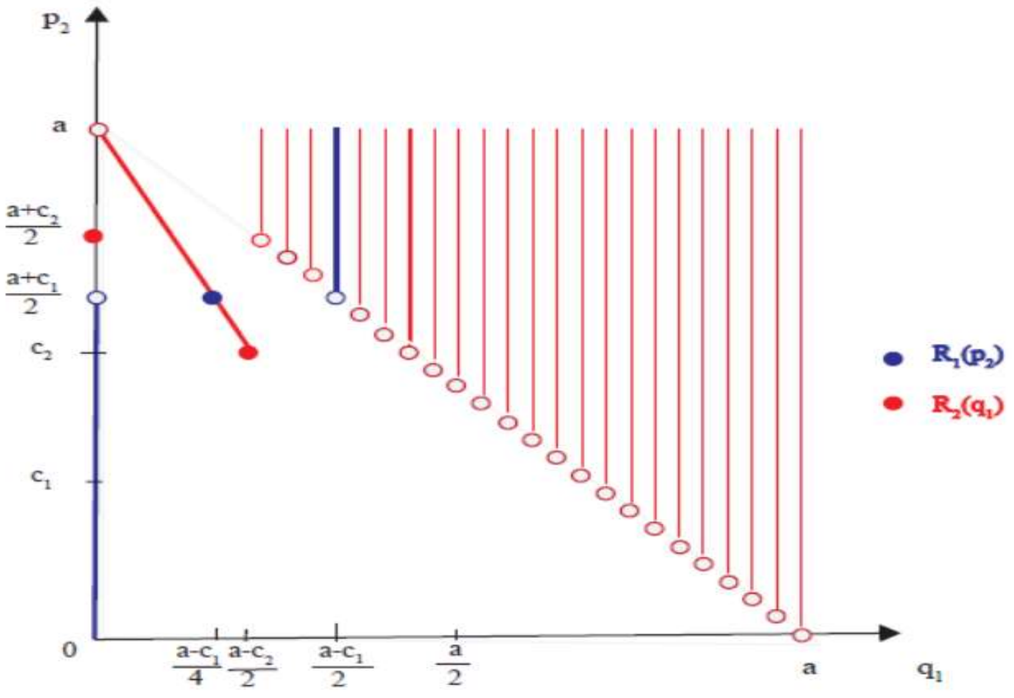

Figure 5

Proof of Proposition 5: Suppose $c_1 = c_2 = c$. First, the Cournot-Bertrand equilibrium $\left(q_C^{CB} = \frac{a-c}{4}, p_B^{CB} = \frac{a+c}{2}\right)$ is unstable under the best-reply dynamics. For any $p_{2,t} > \frac{a+c}{2}$, we have $q_{1,t+1} = R_1\left(p_{2,t}\right) = \frac{a-c}{2}$, and $q_{1,t+1} = R_1\left(p_{2,t}\right) = 0$ for $p_{2,t} < \frac{a+c}{2}$. These imply $\lim_{t\to\infty} R_1\left(p_{2,t}\right) = 0$ or $\frac{a-c}{2}$. Similarly, for any $q_{1,t} < \frac{a-c}{4}$, we have $p_{2,t+1} = R_2\left(q_{1,t}\right) > \frac{3a+c}{4} > \frac{a+c}{2}$. These imply $\lim_{t\to\infty} R_2\left(q_{1,t}\right) \neq \frac{a+c}{2}$. Thus, no neighborhood around $\left(q_C^{CB} = \frac{a-c}{4}, p_B^{CB} = \frac{a+c}{2}\right)$ exists such that the trajectory starting from the neighborhood will converge to it as $t \to \infty$. Second, the Cournot-Bertrand equilibrium $\left(q_C^{CB} = \frac{a-c}{2}, p_B^{CB} > \frac{a+c}{2}\right)$ is unstable as well. For any $p_{2,t} \le \frac{a+c}{2}$, we have $q_{1,t+1} = R_1\left(p_{2,t}\right) = 0 \ne \frac{a-c}{2}$. For any $q_{1,t} \in \left(\frac{a-c}{4}, \frac{a-c}{2}\right)$, we have $p_{2,t+1} = R_2\left(q_{1,t}\right) > \frac{a+c}{2}$. These suggest no neighborhood around $\left(q_C^{CB} = \frac{a-c}{2}, p_B^{CB} > \frac{a+c}{2}\right)$ existing such that the trajectory starting from the neighborhood will converge to it as $t \to \infty$.

Suppose $c_1 > c_2 > 0$. The Cournot-Bertrand equilibrium $\left(q_C^{CB} = 0, p_B^{CB} = \frac{a+c_2}{2}\right)$ is unstable under the best-reply dynamics. For instance, for all $q_{1,t} \in \left(0, \frac{a-c_1}{2}\right)$, we have $p_{2,t+1} = R_2\left(q_{1,t}\right) > \frac{a+c_1}{2} > \frac{a+c_2}{2}$. Thus, no neighborhood around $\left(q_C^{CB} = 0, p_B^{CB} = \frac{a+c_2}{2}\right)$ exists such that the trajectory starting from the neighborhood will converge to it as $t \to \infty$. Similar arguments can be applied to proving that the Cournot-Bertrand equilibria under $c_2 > c_1 > 0$ are unstable as well.