國立臺北大學自然資源與環境管理研究所
107 學年度第二學期『環境系統分析專題』

課程講義(03)：線性規劃回顧與進階主題

Review of Linear Programming and Advanced Topics

- COMPONENTS OF AN OPTIMIZATION MODEL
  - □ Objective Function(s)
    - ⇨ Single vs. Multiple
    - ⇨ Linear vs. Nonlinear
    - ⇨ Convex (Concave) vs. Non-convex
  - □ Constraints
    - ⇨ Constrained vs. Un-constrained
    - ⇨ Linear vs. Nonlinear
    - ⇨ Convex vs. Non-convex Feasible Regions
  - □ Decision Variables
    - ⇨ Continuous vs. Discrete
    - ⇨ Deterministic vs. Stochastic
  - □ System Parameters (Coefficients)
    - ⇨ Deterministic vs. Stochastic
    - ⇨ Division into Sub-Models
  - □ Formulation of Optimization Models
    - ⇨ Plain Form: Straightforward but not suitable for large-scaled or complex problems
    - ⇨ Algebraic Formulations => Parameters (Scalars), Vectors, and Matrices (Tables)
    - ⇨ Algebraic Formulations with text description of variables and parameters
    - ⇨ Sets and Indices => Equation Editor

- PROPERTIES OF AN LP
  - □ Proportionality, Additivity, Divisibility, Certainty , and Non-Negativity
  - □ Non-negative Decision Variables => What if negative values are needed?
  - □ A “Convex Programming” Model
  - □ Additional Terminology
    - ⇨ Feasible Region or Solution Space
    - ⇨ Vertex, Extreme Points or Corner Points
    - ⇨ Decision Space or Objective Space

- SOLUTION PROCEDURE OF AN LP
  - □ Pre-Optimal Analysis, Optimization (Solution) and Post-Optimization Analysis
  - □ Graphical, Simplex, Dual Simplex, Interior Point and Other Methods
  - □ Infeasible, Un-bounded and Degenerate Solutions
  - □ A “Convex Programming” Model: Feasible Region and Extreme Points
    - ⇨ Characteristics of Feasible Region for the LP: Convex, Compact, and Continuous
    - ⇨ Extreme Points (Corner Points) vs. Interior Points

- THE SIMPLEX METHOD
  - Augmented Form of the LP Models
    - ⇨ "Less-than-and-equal-to" Inequality constraints => Slack variables
    - ⇨ "Greater-than-and-equal-to" Inequality constraints => Surplus & Artificial Variables
    - ⇨ Equality constraints => Artificial variables => 'Big-M Treatment'
  - Terminology and Procedure of the Simplex Method
    - ⇨ Basic vs. non-basic variables
    - ⇨ Feasible basic solution => "Adjacent"
    - ⇨ Ratio test for Pivoting
    - ⇨ "Optimality"
  - Simplex Tableaus and An Animated Presentation

- EXAMPLES OF LINEAR PROGRAMMING
  - Homewood Masonry -- A Material Production Problem
    - ⇨ Objective Function: Maximizing the production profit
    - ⇨ Decision Variables: Two building products to be produced
    - ⇨ Constraints: Resource availability, work hours, and curing vat capacity

Resource	HYDIT	FILIT	Availability
Wahash Red Clay	2 m^3/ton	4 m^3/ton	28 m^3/wk
Blending time	5 hr/ton	5 hr/ton	50 hr/wk
Curing vat capacity	8 tons	6 tons	
Profit	\$140/ton	\$160/ton	

1. Algebraic formulation with numerical coefficients

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Maximize Profit $z = 140x_1 + 160x_2$

Subject to

$$2x_1 + 4x_2 \le 28$$
$$5x_1 + 5x_2 \le 50$$
$$x_1 \le 8$$
$$x_2 \le 6$$

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Lingo Code

```
Max=140*x1+160*x2;
  2*x1+4*x2  <=  28;
  5*x1+5*x2  <=  50;
  x1        <=   8;
       x2  <=   6;
```

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2. Algebraic formulation with symbolic coefficients

$$\text{Maximize Profit } z = \sum_{j=1}^{2} c_j x_j$$

Subject to

$$\sum_{j=1}^{2} a_{ij} x_j \le b_i; \quad i = 1, \cdots, 4$$

$$\boldsymbol{c} = \{c_j\} = [140, 160] \quad \boldsymbol{b} = \{b_i\} = \begin{bmatrix} 28 \\ 50 \\ 8 \\ 6 \end{bmatrix} \quad \boldsymbol{A} = \{a_{ij}\} = \begin{bmatrix} 2 & 4 \\ 5 & 5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. Matrix formulation

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Maximize Profit $z = \boldsymbol{c}' \cdot \boldsymbol{x}$

$z = [140, 160] \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Subject to

$\boldsymbol{A} \cdot \boldsymbol{x} \leq \boldsymbol{b}$

$\begin{bmatrix} 2 & 4 \\ 5 & 5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 28 \\ 50 \\ 8 \\ 6 \end{bmatrix}$

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```
MODEL:
  !Title "Homewood Masonry Problem";
 SETS:
  j /1..2/: x, c; !product;
  i /1..4/: b;     !constraint;
  ij(i,j)  : A;    !tech. Coeff.;
 ENDSETS
 DATA:
  c=140 160;
  b=28 50 8 6;
  A=2 4
    5 5
    1 0
    0 1;
 ENDDATA
 MAX= @sum(j: c*x);
 @FOR(i: @sum(j: A*x) <= b );
END
```

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- Sensitivity Analysis
  - □ Overview and Post-Optimality Analysis
  - □ Sensitivity Analysis on RHS (Resource) Coefficients
    - ⇨ Shadow price, marginal value of a resource and economic interpretation
    - ⇨ Dual price (?)
  - □ Sensitivity Analysis on Objective Function Coefficients
  - □ Graphical Illustration
  - □ Outputs from Optimization Packages and Analytical Interpretation (?)
  - □ Parametric Programming

- Duality Theory
  - □ Model Formulations
  - □ Dual-Primal Relationships
    - ⇨ Implementation from Production Problem
    - ⇨ Implementation from Resource Allocation Problem
  - □ Shadow Price
  - □ Primal-Dual Methods for Optimization (Lagrange Algorithms)

- HOMEWORK #2 (2019/03/19 Due)：*Solve* the example problem of Homewood Masonry (ReVelle et al., 2004) by using What'sBest, LINGO, and Euler Math Toolbox.