Chapter 3: Random Variables

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3.1 The Notion of a Random Variable

- A random variable $X$ is a function that assigns a real number, $X(\zeta)$, to each outcome $\zeta$ in the sample space.
Example:

- Toss a coin three times.

\[ S' = \{ \text{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT} \}. \]

- Let \( X \) be the number of heads.

\[ \begin{array}{cccccccc}
\zeta & \text{HHH} & \text{HHT} & \text{HTH} & \text{THH} & \text{HTT} & \text{THT} & \text{TTH} & \text{TTT} \\
X(\zeta) & 3 & 2 & 2 & 2 & 1 & 1 & 1 & 0 \\
\end{array} \]

- \( X \) is a random variable with \( S_X = \{0, 1, 2, 3\} \).
\begin{itemize}
  \item \( S \): Sample space of a random experiment
  \item \( X \): Random variable \( X : S \rightarrow S_X \)
  \item \( S_X \) is a new sample space
  \item Let \( B \subseteq S_X \) and \( A = \{ \zeta : X(\zeta) \in B \} \). Then
    \[ P[B] = P[A] = P[\{ \zeta : X(\zeta) \in B \}] \]
  \item \( A \) and \( B \) are equivalent events.
\end{itemize}
3.2 Cumulative Distribution Function

- Cumulative distribution function (cdf)
  \[ F_X(x) = P[X \leq x] \quad \text{for} \quad -\infty < x < \infty \]

- In underlying sample space
  \[ F_X(x) = P[\{\zeta : X(\zeta) \leq x\}] \]

- \( F_X(x) \) is a function of the variable \( x \).
Properties of cdf

1. $0 \leq F_X(x) \leq 1$.

2. $\lim_{x \to \infty} F_X(x) = 1$.

3. $\lim_{x \to -\infty} F_X(x) = 0$.

4. If $a < b$, then $F_X(a) \leq F_X(b)$.

5. $F_X(x)$ is continuous from the right, i.e., for $h > 0$

   \[ F_X(b) = \lim_{h \to 0} F_X(b + h) = F_X(b^+) . \]

6. $P[a < X \leq b] = F_X(b) - F_X(a)$, since

   \[ \{X \leq a\} \cup \{a < X \leq b\} = \{X \leq b\} . \]
7. $P[X = b] = F_X(b) - F_X(b^-)$.

8. $P[X > x] = 1 - F_X(x)$. 
• $X$: the number of heads in three tosses of a fair coin

• Let $\delta$ be a small positive number. Then

\[ F_X(1-\delta) = P[X \leq 1-\delta] = P\{0 \text{ heads}\} = 1/8, \]
\[ F_X(1) = P[X \leq 1] = P[0 \text{ or } 1 \text{ heads}] = 1/8+3/8 = 1/2, \]
\[ F_X(1+\delta) = P[X \leq 1+\delta] = P[0 \text{ or } 1 \text{ heads}] = 1/2. \]

• Write in unit step function

\[ u(x) = \begin{cases} 
0, & x < 0 \\
1, & x \geq 0, 
\end{cases} \]

\[ F_X(x) = \frac{1}{8}u(x) + \frac{3}{8}u(x-1) + \frac{3}{8}u(x-2) + \frac{1}{8}u(x-3). \]
Example: The transmission time $X$ of messages in a communication system obeys the exponential probability law with parameter $\lambda$, i.e.,

$$P[X > x] = e^{-\lambda x} \quad x > 0$$

Find the cdf of $X$ and $P[T < X \leq 2T]$, where $T = 1/\lambda$.

$$F_X(x) = P[X \leq x] = 1 - P[X > x] = \begin{cases} \ 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0. \end{cases}$$

$$P[T < X \leq 2T] = 1 - e^{-2} - (1 - e^{-1}) = e^{-1} - e^{-2}.$$
Types of random variables

- Discrete random variable, $S_X = x_0, x_1, \ldots$
  \[ F_X(x) = \sum_k p_X(x_k) u(x - x_k), \]
  where $x_k \in S_X$ and $p_X(x_k) = P[X = x_k]$ is the probability mass function (pmf) of $X$.

- Continuous random variable
  \[ F_X(x) = \int_{-\infty}^{x} f(t) dt \]

- Random variable of mixed type
  \[ F_X(x) = pF_1(x) + (1 - p)F_2(x) \]
3.3 Probability Density Function

- Probability density function (pdf) of $X$, if it exists, is

$$f_X(x) = \frac{dF_X(x)}{dx}.$$
Random Variables

\[ P[x < X \leq x + dx] \approx f_X(x)dx \]

(a)

\[ P[a < X < b] = \int_{a}^{b} f_X(x)dx \]

(b)
Properties of pdf

1. $f_X(x) \geq 0$ due to nondecreasing property of cdf.

2. $P[a \leq X \leq b] = \int_a^b f_X(x) \, dx.$

3. $F_X(x) = \int_{-\infty}^x f_X(t) \, dt.$

4. $\int_{-\infty}^\infty f_X(t) \, dt = 1.$
The pdf of the uniform random variable

\[ f_X(x) = \begin{cases} 
\frac{1}{b-a}, & a \leq x \leq b \\
0, & \text{otherwise} 
\end{cases} \]

\[ F_X(x) = \begin{cases} 
0, & x < a \\
\frac{x-a}{b-a}, & a \leq x \leq b \\
1, & x > b 
\end{cases} \]
Pdf for discontinuous cdf

- Unit step function

\[
u(x) = \begin{cases} 
0 & x < 0 \\
1 & x \geq 0 
\end{cases}
\]

- Delta function \( \delta(t) \)

\[
u(x) = \int_{-\infty}^{x} \delta(t)dt
\]

- Cdf for a discrete random variable

\[
F_X(x) = \sum_{k} p_X(x_k)u(x - x_k) = \int_{-\infty}^{x} f_X(t)dt
\]
\[ f_X(x) = \sum_k p_X(x_k) \delta(x - x_k) \]
Conditional cdf’s and pdf’s

- Conditional cdf of $X$ given $A$
  
  $$F_X(x|A) = \frac{P[\{X \leq x\} \cap A]}{P[A]} \quad \text{if } P[A] > 0$$

- Conditional pdf of $X$ given $A$

  $$f_X(x|A) = \frac{d}{dx}F_X(x|A)$$
3.4 Some Important Random Variables

- **Bernoulli random variable**: Let $A$ be an event. The indicator function for $A$ is

$$I_A(\zeta) = \begin{cases} 
0 & \zeta \notin A \\
1 & \zeta \in A
\end{cases}.$$ 

$I_A$ is the Bernoulli random variable. Ex: toss a coin.

- **Binomial random variable**: Let $X$ be the number of times a event $A$ occurs in $n$ independent trials. Let $I_j$ be the indicator function for event $A$ in the $j$th trial. Then

$$X = I_1 + I_2 + \cdots + I_n$$
and

\[ P[X = k] = \binom{n}{k} p^k (1 - p)^{n-k}, \]

where \( p = P[I_j = 1] \).
• Geometric random variable: Count the number $M$ of independent Bernoulli trials until the first success of event $A$.

$$P[M = k] = (1 - p)^{k-1}p \quad k = 1, 2, \ldots,$$

where $p = P[A]$.

• Another version of the geometric random variable is

$$P[k] = (1 - p)^{k}p \quad k = 0, 1, 2, \ldots.$$
- **Exponential random variable**

\[
f_X(x) = \begin{cases} 
0 & x < 0 \\
\lambda e^{-\lambda x} & x \geq 0 
\end{cases}
\]

\[
F_X(x) = \begin{cases} 
0 & x < 0 \\
1 - e^{-\lambda x} & x \geq 0 
\end{cases}
\]

$\lambda$: rate at which events occur.
• The Poisson random variable:
  The pmf is
  \[ P[N = k] = \frac{\alpha^k}{k!} e^{-\alpha} \quad k = 0, 1, 2, \ldots, \]
  where \( \alpha \) is the average number of event occurrences in a specified time interval or region in space.

• The pmf of the Poisson random variable sums to one, since
  \[ \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} e^{-\alpha} = e^{-\alpha} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} = e^{-\alpha} e^\alpha = 1. \]
• **Gaussian (Normal) random variable:**

The pdf is

\[
f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} \quad -\infty < x < \infty,
\]

where \( m \) and \( \sigma \) are real numbers.

The cdf is

\[
P[X \leq x] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{(x'-m)^2}{2\sigma^2}} dx'.
\]

Change variable \( t = (x' - m)/\sigma \) and we have

\[
F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-m)/\sigma} e^{-t^2/2} dt = \Phi \left( \frac{x - m}{\sigma} \right),
\]
where

\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \]
Random Variables

\[ f_X(x) \]
**Q-function** is defined by

\[
Q(x) = 1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt.
\]

Q-function is the probability of “tail” of the pdf.

\[
Q(0) = 1/2 \quad \text{and} \quad Q(-x) = 1 - Q(x).
\]

Q\((x)\) can be obtained by look-up tables.
Example: A communication system accepts a positive voltage $V$ as input and output a voltage $Y = \alpha V + N$, where $\alpha = 10^{-2}$ and $N$ is a Gaussian random variable with parameters $m = 0$ and $\sigma = 2$. Find the value of $V$ that gives $P[Y < 0] = 10^{-6}$.

Sol:

$$P[Y < 0] = P[\alpha V + N < 0] = P[N < -\alpha V] = \Phi \left( \frac{-\alpha V}{\sigma} \right) = Q \left( \frac{\alpha V}{\sigma} \right) = 10^{-6}.$$ 

From the $Q$-function table, we have $\alpha V / \sigma = 4.753$. Thus, $V = (4.753)\sigma / \alpha = 950.6$. 

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3.5 Functions of a Random Variable

• Let $X$ be a random variable. Define another random variable $Y = g(X)$. **Example:** Let the function $h(x) = (x)^+$ be defined as

$$
(x)^+ = \begin{cases} 
0 & x < 0 \\
x & x \geq 0
\end{cases}.
$$

• Let $B = \{x : g(x) \in C\}$. The probability of event $C$ is

$$
P[Y \in C] = P[g(X) \in C] = P[X \in B].
$$

• Three types of equivalent events are useful in determining the cdf and pdf:
1. Discontinuity case: \( \{ g(X) = y_k \} \);
2. cdf: \( \{ g(X) \leq y \} \);
3. pdf: \( \{ y < g(X) \leq y + h \} \).
Example: Let $X$ be a sample voltage of a speech waveform, and suppose that $X$ has a uniform distribution in the interval $[-4d, 4d]$. Let $Y = q(X)$, where the quantizer input-output characteristic is shown below. Find the pmf for $Y$. 
Sol: The event \( \{ Y = q \} \) for \( q \) in \( S_Y \) is equivalent to the event \( \{ X \in I_q \} \), where \( I_q \) is an interval of points mapped into the
representation point $p$. The pmf of $Y$

$$P[Y = q] = \int_{I_q} f_X(t)dt = 1/8 \quad \text{for all } q.$$
**Example:** Let the random variable $Y$ be defined by

$$Y = aX + b,$$

where $a$ is a nonzero constant. Suppose that $X$ has cdf $F_X(x)$, find $F_Y(y)$.

**Sol:** $\{Y \leq y\}$ and $A = \{aX + b \leq y\}$ are equivalent event. If $a > 0$ then $A = \{X \leq (y - b)/a\}$, and thus

$$F_Y(y) = P \left[ X \leq \frac{y - b}{a} \right] = F_X \left( \frac{y - b}{a} \right) \quad a > 0.$$

If $a < 0$, then $A = \{X \geq (y - b)/a\}$ and

$$F_Y(y) = P \left[ X \geq \frac{y - b}{a} \right] = 1 - F_X \left( \frac{y - b}{a} \right).$$
Therefore, we have

\[
f_Y(y) = \begin{cases} 
\frac{1}{a} f_X \left( \frac{y-b}{a} \right) & a > 0 \\
\frac{1}{-a} f_X \left( \frac{y-b}{a} \right) & a < 0 \\
= \frac{1}{|a|} f_X \left( \frac{y-b}{a} \right). & a = 0
\end{cases}
\]
Random Variables

\[ Y = aX + b \]

\[ \{ X \leq \frac{y - b}{a} \} \]

\[ \{ Y \leq y \} \]
**Example:** Let $X$ be a Gaussian random variable with mean $m$ and standard deviation $\sigma$:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{(x-m)^2}{2\sigma^2}} \quad -\infty < x < \infty$$

Let $Y = aX + b$. Find the pdf of $Y$.

**Sol:** From previous example, we have

$$f_Y(y) = \frac{1}{\sqrt{2\pi|a|\sigma}}e^{-\frac{(y-b-am)^2}{2(a\sigma)^2}}.$$

$Y$ also has a Gaussian distribution with mean $am + b$ and standard deviation $|a|\sigma$. 
Example: Let random variable $Y$ be defined by

$$Y = X^2,$$

where $X$ is a continuous random variable. Find the cdf and pdf of $Y$.

Sol: The event $\{Y \leq y\}$ occurs when $\{X^2 \leq y\}$ or equivalently $\{-\sqrt{y} \leq X \leq \sqrt{y}\}$ for $y$ nonnegative. The event is null when $y$ is negative. Then

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ F_X(\sqrt{y}) - F_X(-\sqrt{y}) & y \geq 0 \end{cases}$$

and

$$f_Y(y) = \begin{cases} f_X(\sqrt{y}) - f_X(-\sqrt{y}) & y > 0 \\ \frac{f_X(\sqrt{y})}{2\sqrt{y}} - \frac{f_X(-\sqrt{y})}{-2\sqrt{y}} & y > 0 \end{cases}$$

$$= \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}.$$
• Consider $Y = g(X)$ as shown below

• Consider the event $C_y = \{y < Y < y + dy\}$. Let $B_x$ be its equivalence in the $x$-axis.

• As shown in the figure, $g(x) = y$ has three solutions and

$$B_x = \{x_1 < X < x_1 + dx_1\} \cup \{x_2 < X < x_2 + dx_2\}$$
\[ \cup \{x_3 < X < x_3 + dx_3 \}. \]

Thus,

\[ P[C_y] = f_Y(y) \left| \frac{dy}{dx} \right| = P[B_x] = f_X(x_1) \left| dx_1 \right| + f_X(x_2) \left| dx_2 \right| + f_X(x_3) \left| dx_3 \right|. \]

In general, we have

\[ f_Y(y) = \sum_k f_X(x) \left| \frac{dy}{dx} \right|_{x=x_k} = \sum_k f_X(x) \left. \left| \frac{dx}{dy} \right| \right|_{x=x_k}. \]
Example: Let $Y = X^2$. For $Y \geq 0$, the equation $y = x^2$ has two solutions, $x_0 = \sqrt{y}$ and $x_1 = -\sqrt{y}$. Since $dy/dx = 2x$, we have

$$f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}.$$

Example: Let $Y = \cos(X)$, where $X$ is uniformly distributed in the interval $(0, 2\pi]$. Find the pdf of $Y$. 
**Sol:** Two solutions in the interval, $x_0 = \cos^{-1}(y)$ and $x_1 = 2\pi - x_0$.

\[
\left. \frac{dy}{dx} \right|_{x_0} = -\sin(x_0) = -\sin(\cos^{-1}(y)) = -\sqrt{1-y^2}.
\]
Since \( f_X(x) = 1/(2\pi) \),

\[
f_Y(y) = \frac{1}{2\pi \sqrt{1 - y^2}} + \frac{1}{2\pi \sqrt{1 - y^2}} = \frac{1}{\pi \sqrt{1 - y^2}} \quad \text{for } -1 < y < 1.
\]

The cdf of \( Y \) is

\[
F_Y(y) = \begin{cases} 
0 & y < -1 \\
\frac{1}{2} + \frac{\sin^{-1} y}{\pi} & -1 \leq y \leq 1 \\
1 & y > 1.
\end{cases}
\]
3.6 Expected Value of Random Variables

![Graph showing the expected value of random variables with data points and trial numbers.]
The Expected Value of $X$

- The expected value or mean of a random variable $X$ is defined by

$$E[X] = \int_{-\infty}^{\infty} t f_X(t) dt$$

- If $X$ is a discrete random variable, then

$$E[X] = \sum_{k} x_k p_X(x_k)$$

- Note that $E[X]$ may not converge.
• The mean for a uniform random variable between \( a \) and \( b \) is given by

\[
E[X] = \int_a^b \frac{t}{b-a} dt = \frac{a+b}{2}
\]

\( E[X] \) is the midpoint of the interval \([a, b]\).

• If the pdf of \( X \) is symmetric about a point \( m \), then \( E[X] = m \). That is, when

\[
f_X(m-x) = f_X(m+x),
\]

we have

\[
0 = \int_{-\infty}^{+\infty} (m-t)f_X(t)dt = m - \int_{-\infty}^{+\infty} tf_X(t)dt.
\]
• The pdf of a Gaussian random variable is symmetric at \( x = m \). Therefore, \( E[X] = m \).
Exercise:
Show that if $X$ is a nonnegative random variable, then

$$E[X] = \int_0^\infty (1 - F_X(t)) \, dt$$

if $X$ continuous and nonnegative and

$$E[X] = \sum_{k=0}^\infty P[X > k]$$

if $X$ nonnegative, integer-valued.
Expected value of $Y = g(X)$

- Let $Y = g(X)$, where $X$ is a random variable with pdf $f_X(x)$.
- $Y$ is also a random variable.
- Mean of $Y$ is

$$E[Y] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx.$$
**Variance of X**

- Variance of the random variable $X$ is defined by
  \[
  \text{VAR}[X] = E[(X - E[X])^2].
  \]

- Standard deviation of $X$
  \[
  \text{STD}[X] = \text{VAR}[X]^{1/2} \quad -- \quad \text{measure of the spread of a distribution.}
  \]

- Simplification
  \[
  \text{VAR}[X] = E[X^2] - 2E[X]X + E[X]^2 \\
  = E[X^2] - 2E[X]E[X] + E[X]^2 \\
  = E[X^2] - E[X]^2
  \]
**Example:** Find the variance of the random variable $X$ that is uniformly distributed in the interval $[a, b]$.

$$E[X] = (a + b)/2,$$

and

$$\text{VAR}[X] = \frac{1}{b-a} \int_{a}^{b} \left(x - \frac{a + b}{2}\right)^2 dx$$

Let $y = (x - (a + b)/2)$. Then

$$\text{VAR}[X] = \frac{1}{b-a} \int_{-(b-a)/2}^{(b-a)/2} y^2 dy = \frac{(b-a)^2}{12}.$$
**Example:** Find the variance of a Gaussian random variable.

Multiply the integral of the pdf of $X$ by $\sqrt{2\pi}\sigma$ to obtain

$$\int_{-\infty}^{+\infty} e^{-(x-m)^2/2\sigma^2} \, dx = \sqrt{2\pi}\sigma.$$  

Differentiate both sides with respect to $\sigma$ to get

$$\int_{-\infty}^{+\infty} \left( \frac{(x-m)^2}{\sigma^3} \right) e^{-(x-m)^2/2\sigma^2} \, dx = \sqrt{2\pi}.$$  

Then

$$\text{VAR}[X] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} (x-m)^2 e^{-(x-m)^2/2\sigma^2} \, dx = \sigma^2.$$
• **Properties**

Let $c$ be a constant. Then

\[
\text{VAR}[c] = 0,
\]

\[
\text{VAR}[X + c] = \text{VAR}[X],
\]

\[
\text{VAR}[cX] = c^2 \text{VAR}[X].
\]

• $n$th moment of the random variable $X$ is given by

\[
E[X^n] = \int_{-\infty}^{+\infty} x^n f_X(x) dx.
\]
3.7 Markov and Chebyshev Inequalities

Markov Inequality

- Suppose $X$ is a nonnegative random variable with mean $E[X]$. Then

$$P[X \geq a] \leq \frac{E[X]}{a} \quad \text{for } X \text{ nonnegative}$$

Since

$$E[X] = \int_{0}^{a} t f_X(t) \, dt + \int_{a}^{\infty} t f_X(t) \, dt \geq \int_{a}^{\infty} t f_X(t) \, dt \geq \int_{a}^{\infty} a f_X(t) \, dt = aP[X \geq a].$$
Chebyshev Inequality

- Consider random variable $X$ with $E[X] = m$ and $\text{VAR}[X] = \sigma^2$. Then
  \[ P[|X - m| \geq a] \leq \frac{\sigma^2}{a^2}. \]

- Proof: Let $D^2 = (X - m)^2$. Markov inequality for $D^2$ gives
  \[ P[D^2 \geq a^2] \leq \frac{E[(X - m)^2]}{a^2} = \frac{\sigma^2}{a^2}. \]

- $\{D^2 \geq a^2\}$ and $\{|X - m| \geq a\}$ are equivalent events.
3.9 Transfer Methods

The Characteristic Function

- The characteristic function of a random variable $X$ is defined by

$$
\Phi_X(\omega) = E[e^{j\omega X}]
= \int_{-\infty}^{\infty} f_X(x)e^{j\omega x} \, dx,
$$

where $j = \sqrt{-1}$ is the imaginary unit number.

- $\Phi_X(\omega)$ can be viewed as the expected value of a function of $X$, $e^{j\omega X}$. 
- $\Phi_X(\omega)$ is the Fourier transform of the pdf $f_X(x)$ with a reversal in the sign of the exponent.

- From the Fourier transform inversion formula we have

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_X(\omega)e^{-j\omega x} \, d\omega.$$
**Example:** The characteristic function for an exponentially distributed random variable with parameter $\lambda$ is given by

$$
\Phi_X(\omega) = \int_0^\infty \lambda e^{-\lambda x} e^{j\omega x} \, dx = \int_0^\infty \lambda e^{-(\lambda-j\omega)x} \, dx
$$

$$
= \frac{\lambda}{\lambda - j\omega}.
$$
• If $X$ is a discrete random variable, we have

$$\Phi_X(\omega) = \sum_k p_X(x_k) e^{j\omega x_k}.$$ 

• If $X$ is an integer-valued random variable, we have

$$\Phi_X(\omega) = \sum_{k=-\infty}^{\infty} p_X(k) e^{j\omega k}.$$ 

• The above is the Fourier transform of the sequence $p_X(k)$.

• It is a periodic function of $\omega$ with period $2\pi$.

• By the inversion formula we have

$$p_X(k) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_X(\omega) e^{-j\omega k} \, d\omega \quad k = 0, \pm 1, \pm 2, \ldots.$$
Example: The characteristic function for a geometric random variable is given by

$$\Phi_X(\omega) = \sum_{k=0}^{\infty} pq^k e^{j\omega k} = p \sum_{k=0}^{\infty} (qe^{j\omega})^k$$

$$= \frac{p}{1 - qe^{j\omega}}.$$
• The **moment theorem** states that the moments of $X$ are given by

$$E[X^n] = \frac{1}{j^n} \frac{d^n}{d\omega^n} \Phi_X(\omega) \bigg|_{\omega=0}.$$  

**Proof:** First we expend $e^{j\omega x}$ in power series in the definition of $\Phi_X(\omega)$:

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x) \left\{ 1 + j\omega x + \frac{(j\omega X)^2}{2!} + \cdots \right\} \, dx.$$  

Assuming that all the moments of $X$ are finite and that the series can be integrated term by term, we have

$$\Phi_X(\omega) = 1 + j\omega E[X] + \frac{(j\omega)^2 E[X^2]}{2!} + \cdots$$

$$+ \frac{(j\omega)^n E[X^n]}{n!} + \cdots.$$
If we differentiate $n$ times and evaluate at $\omega = 0$, we have
\[
\frac{d^n}{d\omega^n} \Phi_X(\omega) \bigg|_{\omega=0} = j^n E[X^n].
\]
Example: To find the mean of an exponentially distributed random variable, we differentiate \( \Phi_X(\omega) = \lambda(\lambda - j\omega)^{-1} \) once, and obtain

\[
\Phi'_X(\omega) = \frac{\lambda j}{(\lambda - j\omega)^2}.
\]

Then \( E[X] = \Phi'_X(0)/j = 1/\lambda \).
The Probability Generating Function

• The probability generating function $G_N(z)$ of a nonnegative integer-valued random variable $N$ is defined by

$$G_N(z) = E[z^N] = \sum_{k=0}^{\infty} p_N(k)z^k.$$

• $G_N(z)$ can be viewed as the expected value of a function of $N$, $z^N$.

• $G_N(z)$ is the $z$-transform of the pmf $p_N(k)$ with a sign change in the exponent.

• Similar to the derivation of the moment theorem, we have

$$p_N(k) = \frac{1}{k!} \frac{d^k}{dz^k} G_N(z) \bigg|_{z=0}.$$
\[ \frac{d}{dz} G_N(z) \bigg|_{z=1} = \sum_{k=0}^{\infty} p_N(k) k z^{k-1} \bigg|_{z=1} = \sum_{k=0}^{\infty} k p_N(k) = E[N]. \]

\[ \frac{d^2}{dz^2} G_N(z) \bigg|_{z=1} = \sum_{k=0}^{\infty} p_N(k) k(k-1) z^{k-2} \bigg|_{z=1} \]
\[ = \sum_{k=0}^{\infty} k(k-1) p_N(k) = E[N(N-1)] \]
\[ = E[N^2] - E[N]. \]

• Thus, the mean and variance of \( N \) are given by

\[ E[N] = G'_N(1) \]
and

\[ VAR[N] = G''_N(1) + G'_N(1) - (G'_N(1))^2. \]
Example: The probability generating function for the Poisson random variable with parameter \( \alpha \) is given by

\[
G_N(z) = \sum_{k=0}^{\infty} \frac{\alpha^k z^k}{k!} = e^{-\alpha} \sum_{k=0}^{\infty} \frac{(\alpha z)^k}{k!} = e^{\alpha (z-1)}.
\]

The first two derivatives of \( G_N(z) \) are given by

\[
G_N'(z) = \alpha e^{\alpha (z-1)} \]

and

\[
G_N''(z) = \alpha^2 e^{\alpha (z-1)}.
\]

Therefore,

\[
E[N] = \alpha \quad \text{and} \quad VAR[N] = \alpha^2 + \alpha - \alpha^2 = \alpha.
\]
The Laplace Transform of the pdf

- The Laplace transform of the pdf is given by

\[ X^*(s) = \int_0^\infty f_X(x)e^{-sx} \, dx = E[e^{-sX}] \cdot \]

- \( X^*(s) \) can be viewed as an expected value of a function of \( X \), \( e^{-sX} \).

- The moment theorem also holds for \( X^*(s) \):

\[ E[X^n] = (-1)^n \frac{d^n}{ds^n}X^*(s) \bigg|_{s=0} \cdot \]