1.1 THE CARTESIAN PLANE AND THE DISTANCE FORMULA

- Plot points in a coordinate plane and read data presented graphically.
- Find the distance between two points in a coordinate plane.
- Find the midpoints of line segments connecting two points.
- Translate points in a coordinate plane.

The Cartesian Plane

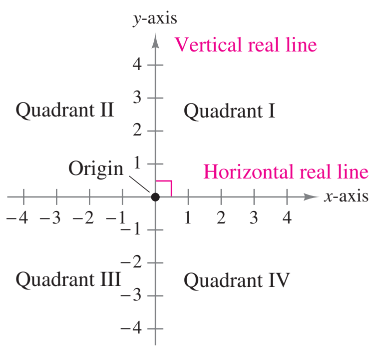
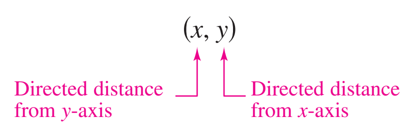


FIGURE 1.1 The Cartesian Plane

Just as you can represent real numbers by points on a real number line, you can represent ordered pairs of real numbers by points in a plane called the **rectangular coordinate system,** or the **Cartesian plane,** after the French mathematician René Descartes (1596–1650).

The Cartesian plane is formed by using two real number lines intersecting at right angles, as shown in Figure 1.1. The horizontal real number line is usually called the ***x*-axis,** and the vertical real number line is usually called the ***y*-axis.** The point of intersection of these two axes is the **origin,** and the two axes divide the plane into four parts called **quadrants.**

Each point in the plane corresponds to an **ordered pair** (x, y) of real numbers x and y, called **coordinates** of the point. The ***x*-coordinate** represents the directed distance from the y-axis to the point, and the ***y*-coordinate** represents the directed distance from the x-axis to the point, as shown in Figure 1.2.



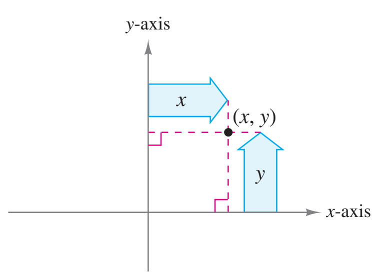


FIGURE 1.2

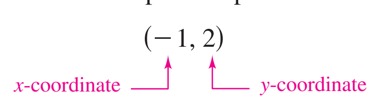
STUDY TIP

The notation (x, y) denotes both a point in the plane and an open interval on the real number line. The context will tell you which meaning is intended.

EXAMPLE 1 Plotting Points in the Cartesian Plane

Plot the points $(-1, 2)$, $(3, 4)$, $(0, 0)$, $(3, 0)$, and $(-2, -3)$.

SOLUTION To plot the point



imagine a vertical line through -1 on the x-axis and a horizontal line through 2 on the y-axis. The intersection of these two lines is the point $(-1, 2)$. The other four points can be plotted in a similar way and are shown in Figure 1.3.

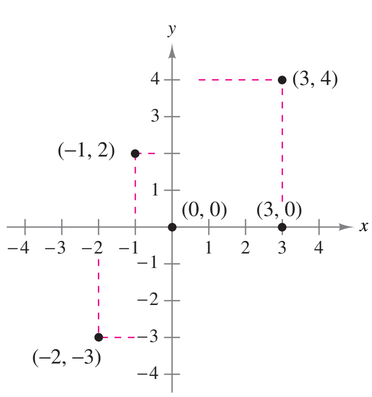


FIGURE 1.3

TRY IT 1

Plot the points $(-3, 2)$, $(4, -2)$, $(3, 1)$, $(0, -2)$, and $(-1, -2)$.

Using a rectangular coordinate system allows you to visualize relationships between two variables. It would be difficult to overestimate the importance of Descartes's introduction of coordinates to the plane. Today his ideas are in common use in virtually every scientific and business-related field. In Example 2, notice how much your intuition is enhanced by the use of a graphical presentation.

EXAMPLE 2 Sketching a Scatter Plot

The amounts A (in millions of dollars) spent on snowmobiles in the United States from 1993 through 2002 are shown in the table, where t represents the year. Sketch a scatter plot of the data. *(Source: National Sporting Goods Association)*

t	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002
A	515	715	910	974	975	883	820	894	784	808

SOLUTION To sketch a *scatter plot* of the data given in the table, you simply represent each pair of values by an ordered pair (t, A), and plot the resulting points, as shown in Figure 1.4. For instance, the first pair of values is represented by the ordered pair (1993, 515). Note that the break in the t-axis indicates that the numbers between 0 and 1992 have been omitted.

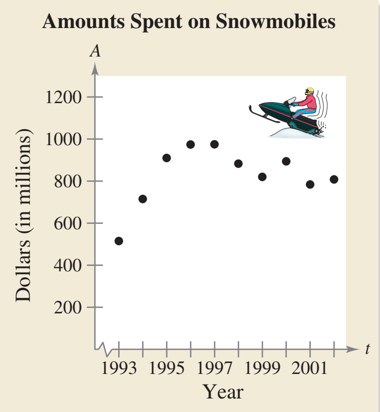


FIGURE 1.4

TRY IT 2

From 1991 to 2000, the enrollments E (in millions) of students in U.S. public colleges are shown, where t represents the year. Sketch a scatter plot of the data. *(Source: U.S. National Center for Education Statistics)*

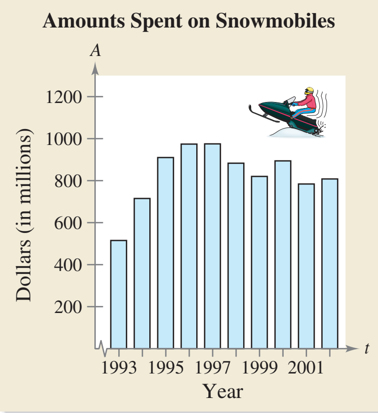
t	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
E	11.3	11.4	11.2	11.1	11.1	11.1	11.2	11.1	11.3	11.8

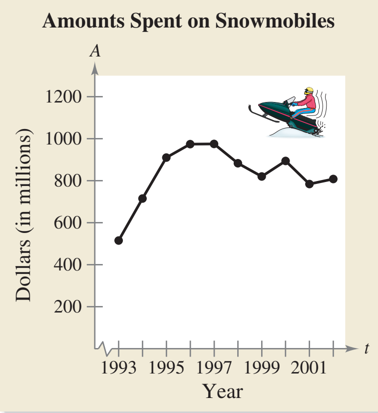
STUDY TIP

In Example 2, you could let $t = 1$ represent the year 1993. In that case, the horizontal axis would not have been broken, and the tick marks would have been labeled 1 through 10 (instead of 1993 through 2002).

TECHNOLOGY

The scatter plot in Example 2 is only one way to represent the given data graphically. Two other techniques are shown at the right. The first is a *bar graph* and the second is a *line graph*. All three graphical representations were created with a computer. If you have access to computer graphing software, try using it to represent graphically the data given in Example 2.





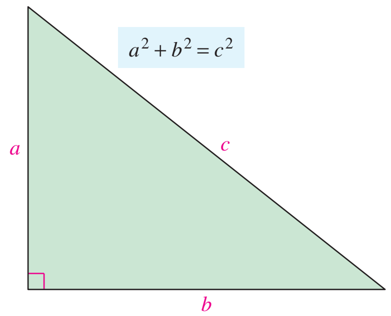


FIGURE 1.5 Pythagorean Theorem

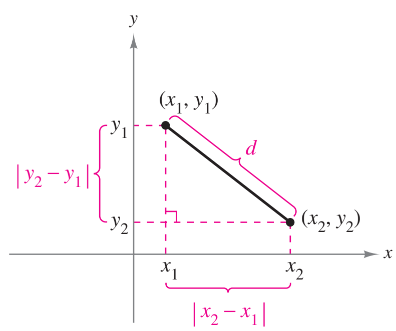


FIGURE 1.6 Distance Between Two Points

The Distance Formula

Recall from the Pythagorean Theorem that, for a right triangle with hypotenuse of length c and sides of lengths a and b, you have

$$a^2 + b^2 = c^2 \qquad \text{Pythagorean Theorem}$$

as shown in Figure 1.5. (The converse is also true. That is, if $a^2 + b^2 = c^2$, then the triangle is a right triangle.)

Suppose you want to determine the distance d between two points (x_1, y_1) and (x_2, y_2) in the plane. With these two points, a right triangle can be formed, as shown in Figure 1.6. The length of the vertical side of the triangle is

$$|y_2 - y_1|$$

and the length of the horizontal side is

$$|x_2 - x_1|.$$

By the Pythagorean Theorem, you can write

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$
$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This result is the **Distance Formula.**

The Distance Formula

The distance d between the points (x_1, y_1) and (x_2, y_2) in the plane is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLE 3 Finding a Distance

Find the distance between the points $(-2, 1)$ and $(3, 4)$.

SOLUTION Let $(x_1, y_1) = (-2, 1)$ and $(x_2, y_2) = (3, 4)$. Then apply the Distance Formula as shown.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{Distance Formula} \\ &= \sqrt{[3 - (-2)]^2 + (4 - 1)^2} && \text{Substitute for } x_1, y_1, x_2, \text{ and } y_2. \\ &= \sqrt{(5)^2 + (3)^2} && \text{Simplify.} \\ &= \sqrt{34} \\ &\approx 5.83 && \text{Use a calculator.} \end{aligned}$$

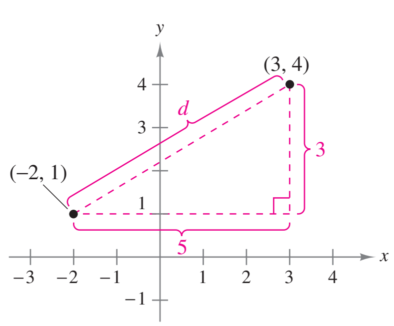


FIGURE 1.7

Note in Figure 1.7 that a distance of 5.83 looks about right.

TRY IT 3

Find the distance between the points $(-2, 1)$ and $(2, 4)$.

EXAMPLE 4 Verifying a Right Triangle

Use the Distance Formula to show that the points (2, 1), (4, 0), and (5, 7) are vertices of a right triangle.

SOLUTION The three points are plotted in Figure 1.8. Using the Distance Formula, you can find the lengths of the three sides as shown below.

$$d_1 = \sqrt{(5-2)^2 + (7-1)^2} = \sqrt{9+36} = \sqrt{45}$$
$$d_2 = \sqrt{(4-2)^2 + (0-1)^2} = \sqrt{4+1} = \sqrt{5}$$
$$d_3 = \sqrt{(5-4)^2 + (7-0)^2} = \sqrt{1+49} = \sqrt{50}$$

Because

$$d_1^2 + d_2^2 = 45 + 5 = 50 = d_3^2$$

you can apply the converse of the Pythagorean Theorem to conclude that the triangle must be a right triangle.

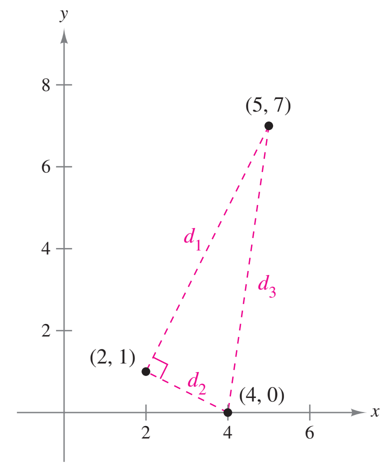


FIGURE 1.8

TRY IT 4

Use the Distance Formula to show that the points (2, −1), (5, 5), and (6, −3) are vertices of a right triangle.

The figures provided with Examples 3 and 4 were not really essential to the solution. *Nevertheless*, we strongly recommend that you develop the habit of including sketches with your solutions—even if they are not required.

EXAMPLE 5 Finding the Length of a Pass

In a football game, a quarterback throws a pass from the five-yard line, 20 yards from the sideline. The pass is caught by a wide receiver on the 45-yard line, 50 yards from the same sideline, as shown in Figure 1.9. How long was the pass?

SOLUTION You can find the length of the pass by finding the distance between the points (20, 5) and (50, 45).

$$\begin{aligned} d &= \sqrt{(50-20)^2 + (45-5)^2} && \text{Distance Formula} \\ &= \sqrt{900 + 1600} \\ &= 50 && \text{Simplify.} \end{aligned}$$

So, the pass was 50 yards long.

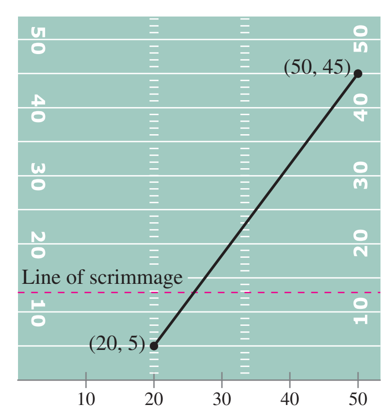


FIGURE 1.9

STUDY TIP

In Example 5, the scale along the goal line showing distance from the sideline does not normally appear on a football field. However, when you use coordinate geometry to solve real-life problems, you are free to place the coordinate system in any way that is convenient to the solution of the problem.

TRY IT 5

A quarterback throws a pass from the 10-yard line, 10 yards from the sideline. The pass is caught by a wide receiver on the 30-yard line, 25 yards from the same sideline. How long was the pass?

The Midpoint Formula

To find the **midpoint** of the line segment that joins two points in a coordinate plane, you can simply find the average values of the respective coordinates of the two endpoints.

The Midpoint Formula

The midpoint of the segment joining the points (x_1, y_1) and (x_2, y_2) is

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

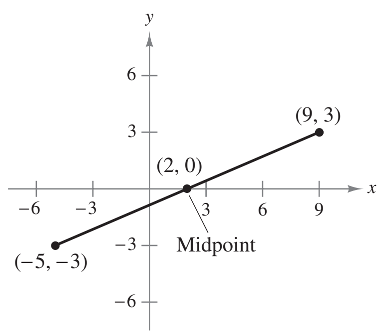


FIGURE 1.10

EXAMPLE 6 Finding a Segment's Midpoint

Find the midpoint of the line segment joining the points $(-5, -3)$ and $(9, 3)$, as shown in Figure 1.10.

SOLUTION Let $(x_1, y_1) = (-5, -3)$ and $(x_2, y_2) = (9, 3)$.

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-5 + 9}{2}, \frac{-3 + 3}{2}\right) = (2, 0)$$

TRY IT 6

Find the midpoint of the line segment joining $(-6, 2)$ and $(2, 8)$.

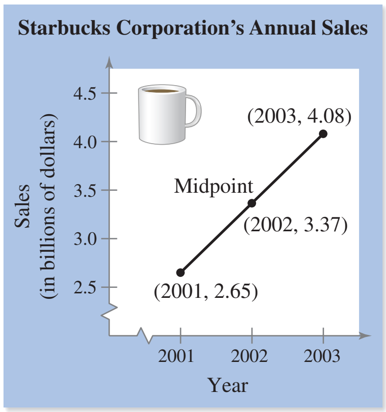


FIGURE 1.11

EXAMPLE 7 Estimating Annual Sales

Starbucks Corporation had annual sales of \$2.65 billion in 2001 and \$4.08 billion in 2003. Without knowing any additional information, what would you estimate the 2002 sales to have been? *(Source: Starbucks Corp.)*

SOLUTION One solution to the problem is to assume that sales followed a linear pattern. With this assumption, you can estimate the 2002 sales by finding the midpoint of the segment connecting the points (2001, 2.65) and (2003, 4.08).

$$\text{Midpoint} = \left(\frac{2001 + 2003}{2}, \frac{2.65 + 4.08}{2}\right) \approx (2002, 3.37)$$

So, you would estimate the 2002 sales to have been about \$3.37 billion, as shown in Figure 1.11. (The actual 2002 sales were \$3.29 billion.)

TRY IT 7

Maytag Corporation had annual sales of \$4.32 billion in 2001 and \$4.79 billion in 2003. What would you estimate the 2002 annual sales to have been? *(Source: Maytag Corp.)*

Translating Points in the Plane

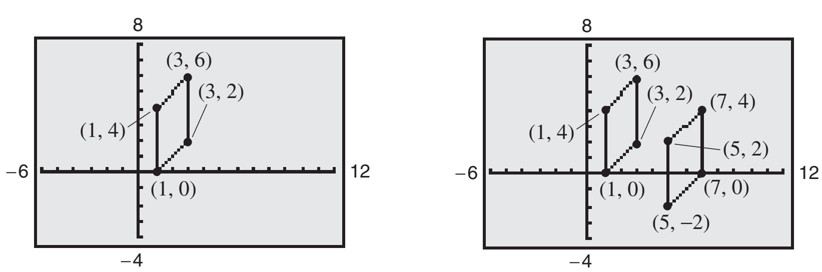
EXAMPLE 8 Translating Points in the Plane

Figure 1.12(a) shows the vertices of a parallelogram. Find the vertices of the parallelogram after it has been translated two units down and four units to the right.

SOLUTION To translate each vertex two units down, subtract 2 from each y-coordinate. To translate each vertex four units to the right, add 4 to each x-coordinate.

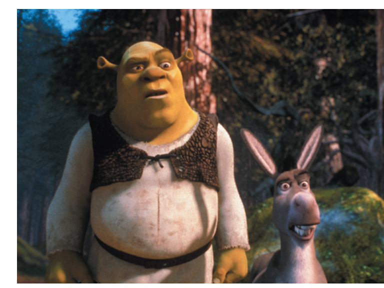
Original Point	*Translated Point*
$(1, 0)$	$(1 + 4, 0 - 2) = (5, -2)$
$(3, 2)$	$(3 + 4, 2 - 2) = (7, 0)$
$(3, 6)$	$(3 + 4, 6 - 2) = (7, 4)$
$(1, 4)$	$(1 + 4, 4 - 2) = (5, 2)$

The translated parallelogram is shown in Figure 1.12(b).



(a) (b)

FIGURE 1.12



DREAMWORKS/THE KOBAL COLLECTION

Many movies now use extensive computer graphics, much of which consists of transformations of points in two- and three-dimensional space. The photo above shows a scene from Shrek. *The movie's animators used computer graphics to design the scenery, characters, motion, and even the lighting in each scene.*

TRY IT 8

Find the vertices of the parallelogram in Example 8 after it has been translated two units to the left and four units down.

TAKE ANOTHER LOOK

Transforming Points in a Coordinate Plane

Example 8 illustrates points that have been translated (or *slid*) in a coordinate plane. The translated parallelogram is congruent to (has the same size and shape as) the original parallelogram. Try using a graphing utility to graph the transformed parallelogram for each of the following transformations. Describe the transformation. Is it a translation, a reflection, or a rotation? Is the transformed parallelogram congruent to the original parallelogram?

a. $(x, y) \Rightarrow (-x, y)$

b. $(x, y) \Rightarrow (x, -y)$

c. $(x, y) \Rightarrow (-x, -y)$

PREREQUISITE REVIEW 1.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–6, simplify each expression.

1. $\sqrt{(3-6)^2 + [1-(-5)]^2}$

2. $\sqrt{(-2-0)^2 + [-7-(-3)]^2}$

3. $\dfrac{5+(-4)}{2}$

4. $\dfrac{-3+(-1)}{2}$

5. $\sqrt{27} + \sqrt{12}$

6. $\sqrt{8} - \sqrt{18}$

In Exercises 7–10, solve for x or y.

7. $\sqrt{(3-x)^2 + (7-4)^2} = \sqrt{45}$

8. $\sqrt{(6-2)^2 + (-2-y)^2} = \sqrt{52}$

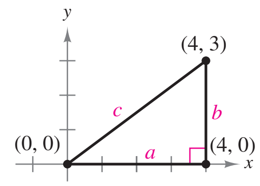
9. $\dfrac{x+(-5)}{2} = 7$

10. $\dfrac{-7+y}{2} = -3$

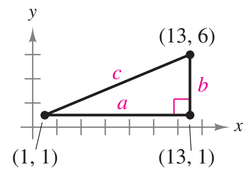
EXERCISES 1.1

In Exercises 1–6, (a) find the length of each side of the right triangle and (b) show that these lengths satisfy the Pythagorean Theorem.

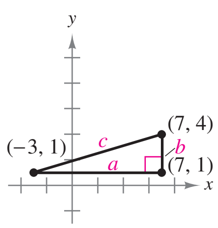
***1.**



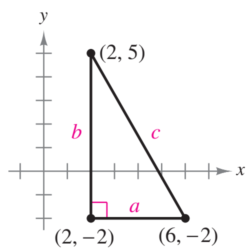
2.



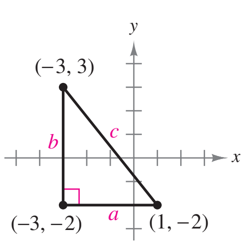
3.



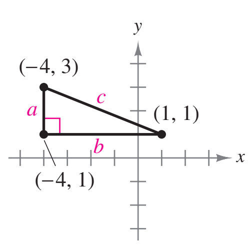
4.



5.



6.



* The answers to the odd-numbered and selected even exercises are given in the back of the text. Worked-out solutions to the odd-numbered exercises are given in the *Student Solutions Guide*.

In Exercises 7–14, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.

7. $(3, 1), (5, 5)$

8. $(-3, 2), (3, -2)$

9. $\left(\frac{1}{2}, 1\right), \left(-\frac{3}{2}, -5\right)$

10. $\left(\frac{2}{3}, -\frac{1}{3}\right), \left(\frac{5}{6}, 1\right)$

11. $(2, 2), (4, 14)$

12. $(-3, 7), (1, -1)$

13. $\left(1, \sqrt{3}\right), (-1, 1)$

14. $(-2, 0), \left(0, \sqrt{2}\right)$

In Exercises 15–18, show that the points form the vertices of the given figure. (A rhombus is a quadrilateral whose sides have the same length.)

	Vertices	*Figure*
15.	$(0, 1), (3, 7), (4, -1)$	Right triangle
16.	$(1, -3), (3, 2), (-2, 4)$	Isosceles triangle
17.	$(0, 0), (1, 2), (2, 1), (3, 3)$	Rhombus
18.	$(0, 1), (3, 7), (4, 4), (1, -2)$	Parallelogram

In Exercises 19–22, use the Distance Formula to determine whether the points are collinear (lie on the same line).

19. $(0, -4), (2, 0), (3, 2)$

20. $(0, 4), (7, -6), (-5, 11)$

21. $(-2, -6), (1, -3), (5, 2)$

22. $(-1, 1), (3, 3), (5, 5)$

In Exercises 23 and 24, find x such that the distance between the points is 5.

23. $(1, 0), (x, -4)$

24. $(2, -1), (x, 2)$

In Exercises 25 and 26, find y such that the distance between the points is 8.

25. $(0, 0), (3, y)$

26. $(5, 1), (5, y)$

27. Use the Midpoint Formula repeatedly to find the three points that divide the segment joining (x_1, y_1) and (x_2, y_2) into four equal parts.

28. Show that $\left(\frac{1}{3}[2x_1 + x_2], \frac{1}{3}[2y_1 + y_2]\right)$ is one of the points of trisection of the line segment joining (x_1, y_1) and (x_2, y_2). Then, find the second point of trisection by finding the midpoint of the segment joining

$$\left(\frac{1}{3}[2x_1 + x_2], \frac{1}{3}[2y_1 + y_2]\right) \text{ and } (x_2, y_2).$$

29. Use Exercise 27 to find the points that divide the line segment joining the given points into four equal parts.

(a) $(1, -2), (4, -1)$

(b) $(-2, -3), (0, 0)$

30. Use Exercise 28 to find the points of trisection of the line segment joining the given points.

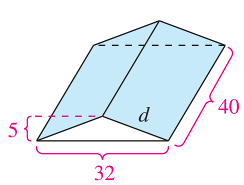
(a) $(1, -2), (4, 1)$

(b) $(-2, -3), (0, 0)$

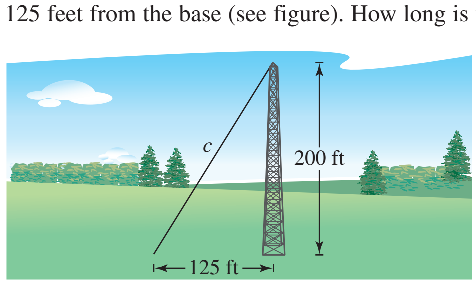
31. ***Building Dimensions*** The base and height of the trusses for the roof of a house are 32 feet and 5 feet, respectively (see figure).

(a) Find the distance d from the eaves to the peak of the roof.

(b) The length of the house is 40 feet. Use the result of part (a) to find the number of square feet of roofing.



32. ***Wire Length*** A guy wire is stretched from a broadcasting tower at a point 200 feet above the ground to an anchor 125 feet from the base (see figure). How long is the wire?



The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system. The solutions of other exercises may also be facilitated by use of appropriate technology.

In Exercises 33 and 34, use a graphing utility to graph a scatter plot, a bar graph, or a line graph to represent the data. Describe any trends that appear.

33. ***Consumer Trends*** The numbers (in millions) of cable television subscribers in the United States for 1992–2001 are shown in the table. *(Source: Nielsen Media Research)*

Year	1992	1993	1994	1995	1996
Subscribers	57.2	58.8	60.5	63.0	64.6

Year	1997	1998	1999	2000	2001
Subscribers	65.9	67.0	68.5	69.3	73.0

34. ***Consumer Trends*** The numbers (in millions) of cellular telephone subscribers in the United States for 1993–2002 are shown in the table. *(Source: Cellular Telecommunications & Internet Association)*

Year	1993	1994	1995	1996	1997
Subscribers	16.0	24.1	33.8	44.0	55.3

Year	1998	1999	2000	2001	2002
Subscribers	69.2	86.0	109.5	128.4	140.8

Dow Jones Industrial Average In Exercises 35 and 36, use the figure below showing the Dow Jones Industrial Average for common stocks. *(Source: Dow Jones, Inc.)*

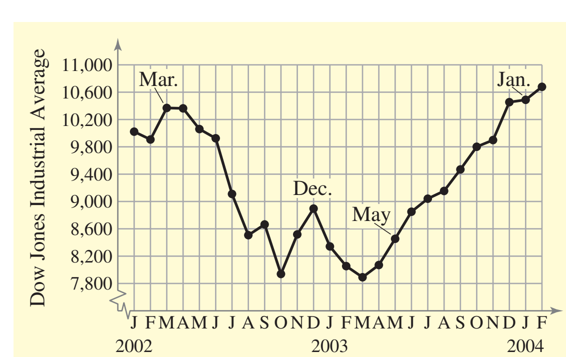
35. Estimate the Dow Jones Industrial Average for each date.

(a) March 2002 (b) December 2002

(c) May 2003 (d) January 2004

36. Estimate the percent increase or decrease in the Dow Jones Industrial Average (a) from April 2002 to November 2002 and (b) from June 2003 to February 2004.

Figure for 35 and 36



Construction In Exercises 37 and 38, use the figure, which shows the median sales prices of existing one-family homes sold (in thousands of dollars) in the United States from 1987 to 2002. *(Source: National Association of Realtors)*

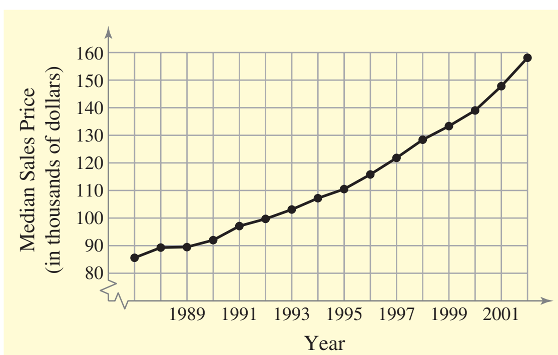
37. Estimate the median sales price of existing one-family homes for each year.

(a) 1987 (b) 1992

(c) 1997 (d) 2002

38. Estimate the percent increases in the value of existing one-family homes (a) from 1993 to 1994 and (b) from 2001 to 2002.

Figure for 37 and 38



Research Project In Exercises 39 and 40, (a) use the Midpoint Formula to estimate the revenue and profit of the company in 2001. (b) Then use your school's library, the Internet, or some other reference source to find the actual revenue and profit for 2001. (c) Did the revenue and profit increase in a linear pattern from 1999 to 2003? Explain your reasoning. (d) What were the company's expenses during each of the given years? (e) How would you rate the company's growth from 1999 to 2003? *(Source: Walgreen Company and The Yankee Candle Company)*

39. Walgreen Company

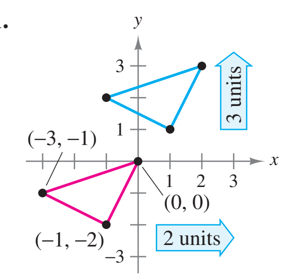
Year	1999	2001	2003
Revenue (millions of \$)	17,839		32,505
Profit (millions of \$)	624.1		1157.3

40. The Yankee Candle Company

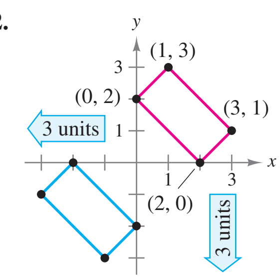
Year	1999	2001	2003
Revenue (millions of \$)	256.6		508.6
Profit (millions of \$)	34.3		74.8

Computer Graphics In Exercises 41 and 42, the red figure is translated to a new position in the plane to form the blue figure. (a) Find the vertices of the transformed figure. (b) Then use a graphing utility to draw both figures.

41.



42.



43. ***Economics*** The table shows the numbers of ear infections treated by doctors at HMO clinics of three different sizes: small, medium, and large.

Cases per small clinic	Cases per medium clinic	Cases per large clinic	Number of doctors
0	0	0	0
20	30	35	1
28	42	49	2
35	53	62	3
40	60	70	4

(a) Show the relationship between doctors and treated ear infections using *three* curves, where the number of doctors is on the horizontal axis and the number of ear infections treated is on the vertical axis.

(b) Compare the three relationships.

(Source: Adapted from Taylor, Economics, *Fourth Edition)*

The symbol indicates an exercise that contains material from textbooks in other disciplines.