One-Sample Test for Proportion

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February 26, 2006

Approximated One-Sample Z Test for Proportion

DM-TKA Example

In DM-TKR Data, there are 5 infective patients of total 78 patients, the sample proportion is $6.41\% = 5/78$. The infectiive probability in U.S. is about 1%. Does our sample differ from U.S. population?

DM-TKA Example

1. π be infective probability in population

2. Random sample of n observations

3. X_i be random variables for each individual, $i = 1, \ldots n$,

$$X_i = \begin{cases} 1, & \text{infection with probability } \pi, \\ 0, & \text{no infection with probability } 1 - \pi. \end{cases} \tag{1}$$

4. $Y = \sum_{i=1}^{n} X_i$ has Binomial distribution (n, π)

There is quite a variety of hypotheses about the DM population infective probability π.

Hypothesis

- $H_0 : \pi = \pi_0 = 0.01,$
- $H_A : \pi \neq \pi_0.$

Test Statistics

The observable sample proportion

$$\hat{\pi} = \frac{Y}{n} = \frac{\sum_{i=1}^{n} X_i}{n}, \tag{2}$$

The sample distribution of the sample proportion $\hat{\pi}$

$$\hat{\pi} \sim N\left(\pi, \frac{\sqrt{\pi(1-\pi)}}{\sqrt{n}}\right) \tag{3}$$

The obsereved sample test statistic under H_0

$$\mathsf{Z} = \frac{(\hat{\pi} - \pi_0)}{\sqrt{(\pi_0(1-\pi_0)/n)}} \sim N(0,1) \tag{4}$$

(approximate distribution)

(It is called approximated Z test since it use the Z statistic.)

Testing Hypothesis: Z Value Method

The test statistic Z depends upon

1. The sample proportion $\hat{\pi}$

2. The hypothesized target general population proportion π

3. The population standard deviation, $\sqrt{\pi(1-\pi)}$.

If the null hypothesis H_0 is true, then the hypothesized population proportion $\pi_0 = 0.01$ is equal to the population proportion, π.

Testing Hypothesis: Z Value Method

1. Prescirbe Type I Error α

2. $Z_{1-\alpha/2}$ be the corresponding percentile from $N(0,1)$ such tat $P(Z < Z_\alpha) = \alpha$

3. Under $H_0 : \pi = \pi_0$, the observed test statistic

$$\mathsf{z} = \frac{(\hat{\pi} - \pi_0)}{\sqrt{\pi_0(1-\pi_0)}/\sqrt{n}}. \tag{5}$$

Testing Hypothesis: Z Value Method

1. For two-sided alternative test, $H_A : \pi \neq \pi_0$

2. Reject the H_0 when

$$|\mathsf{z}| > Z_{1-\alpha/2}.$$

Critical Value and Critical Region Methods

Given the significant level α

$$P(|\mathsf{Z}| > Z_{1-\alpha/2}) = \alpha$$

$$P\Big(\Big|\frac{(\hat{\pi} - \pi)}{\sqrt{\pi(1-\pi)}/\sqrt{n}}\Big| > Z_{1-\alpha/2}\Big) = \alpha$$

$$P\Big(\hat{\pi} < \pi - Z_{1-\alpha/2}\frac{\sqrt{\pi(1-\pi)}}{\sqrt{n}} \quad \text{or} \quad \hat{\pi} > \pi + Z_{1-\alpha/2}\frac{\sqrt{\pi(1-\pi)}}{\sqrt{n}}\Big)$$

Critical Value and Critical Region Methods

Under $H_0 : \pi = \pi_0$, we choose the two **critical values** for the two-sided Z test are

$$c_{\alpha,1} = \pi_0 - Z_{1-\alpha/2}\frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{n}} \tag{6}$$

and $$c_{\alpha,2} = \pi_0 + Z_{1-\alpha/2}\frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{n}}. \tag{7}$$

We will reject the H_0 based on the **critical region** when $\hat{\pi} = \frac{y}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$

$$\hat{\pi} < c_{\alpha,1} = \pi_0 - Z_{1-\alpha/2}\frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{n}}. \tag{8}$$

or $$\hat{\pi} > c_{\alpha,2} = \pi_0 + Z_{1-\alpha/2}\frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{n}}. \tag{9}$$

Confidence Interval Method

The two-sided $(1-\alpha)\times 100\%$ confidence interval of the population proportion π based on the sample statistic $\hat{\pi}$, and the two-sided alternative hypothesis $H_A: \pi \neq \pi_0$, is

$$P[|\mathsf{Z}| < Z_{1-\alpha/2}] = 1-\alpha$$

$$P\left[\left|\frac{\hat{\pi}-\pi}{\sqrt{\pi(1-\pi)}/\sqrt{n}}\right| < Z_{1-\alpha/2}\right] = 1-\alpha$$

$$P\left[\left|(\hat{\pi}-\pi)\right| < Z_{1-\alpha/2} \times \frac{\sqrt{\pi(1-\pi)}}{\sqrt{n}}\right] = 1-\alpha$$

$$P\left[\pi > \hat{\pi} - Z_{1-\alpha/2} \times \frac{\sqrt{\pi(1-\pi)}}{\sqrt{n}}\right.$$

$$\left.\text{and} \quad \pi < \hat{\pi} + Z_{1-\alpha/2} \times \frac{\sqrt{\pi(1-\pi)}}{\sqrt{n}}\right] = 1-\alpha.$$

Asymptotic Confidence Interval Method

The two-sided $(1-\alpha) \times 100\%$ confidence interval of the population proportion π based on the sample statistic $\hat{\pi}$, is

$$\left(\hat{\pi} - Z_{1-\alpha/2} \times \frac{\sqrt{\hat{\pi}(1-\hat{\pi})}}{\sqrt{n}},\ \hat{\pi} + Z_{1-\alpha/2} \times \frac{\sqrt{\hat{\pi}(1-\hat{\pi})}}{\sqrt{n}}\right). \tag{10}$$

We will reject the two-sided test when the two-sided $(1-\alpha) \times 100\%$ confidence interval of the population does not contain the hypothesized population proportion π_0 under H_0.

Asymptotic Confidence Interval Method

For $H_0 : \pi = \pi_0$ versus $H_A : \pi \neq \pi_0$, we will reject the H_0 when

$$\pi_0 < \hat{\pi} - Z_{1-\alpha/2} \times \frac{\sqrt{\hat{\pi}(1-\hat{\pi})}}{\sqrt{n}}, \tag{11}$$

or

$$\pi_0 > \hat{\pi} + Z_{1-\alpha/2} \times \frac{\sqrt{\hat{\pi}(1-\hat{\pi})}}{\sqrt{n}}. \tag{12}$$

That is when the hypothesized proportion π_0 is below the lower or above the upper confident limit, we will reject H_0.

Score (Wilson) Confidence Interval Method

For $H_0 : \pi = \pi_0$ versus $H_A : \pi \neq \pi_0$, we can calculate the C.I. based on the **score confidence interval** (**Wilson's confidence interval**) as

$$\left[\hat{\pi} - Z_{1-\alpha/2} \times \frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{n}}\ \hat{\pi} + Z_{1-\alpha/2} \times \frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{n}}\right]. \quad (13)$$

Likelihood Confidence Interval Method

1. For a sample proportion $\hat{\pi}$ based on n, we can find the value π_{0L} and π_{0R} for the null hypothesis parameters that lead to the test statistic value z

$$\frac{|\hat{\pi} - \pi_0|}{\sqrt{\pi_0(1-\pi_0)/n}} = 1.96 \tag{14}$$

and a two-sided $p-$value $= 0.05$.

2. By solving the equation for π_0, this equation is quadratic in π_0.

3. Alternative, one can determine the limits by trial and error.

p-**Value Method**

1. We have collected the data and the observed sample statistic is $\hat{\pi}$.

2. Consider the two-sided hypothesis
   $H_0 : \pi = \pi_0$ versus $H_A : \pi \neq \pi_0$.

3. The observed two-sided Z test sample statistic is

$$\text{z} = \frac{(\hat{\pi} - \pi_0)}{(\sqrt{\pi_0(1 - \pi_0)}/\sqrt{n})}. \tag{15}$$

4. The p-value is defined as

   - The p-value is the probability of obtaining a result as/or more extreme than you did by chance alone assuming the null hypothesis H_0 is true.

p-Value Method

The p-value for two-sided test is calculated as

$$
\begin{aligned}
& P(|\overline{Y}| > |\bar{y}| \;\mid\; \pi = \pi_0) \\
= \;& P\Big(|\pi - \pi_0| > |\hat{\pi} - \pi_0| \;\mid\; \pi = \pi_0\Big) \\
= \;& P\Big(\Big|\frac{(\pi - \pi_0)}{\sqrt{\pi_0(1-\pi_0)}/\sqrt{n}}\Big| > \Big|\frac{(\hat{\pi} - \pi_0)}{\sqrt{\pi_0(1-\pi_0)}/\sqrt{n}}\Big| \;\mid\; \pi = \pi_0\Big) \\
= \;& P(|\mathsf{Z}| > |\mathsf{z}| \;\mid\; \pi = \pi_0) \\
= \;& 2[1 - P(\mathsf{Z} \le |\mathsf{z}| \;\mid\; \pi = \pi_0)] \\
= \;& 2[1 - \Phi(|\mathsf{z}|)],
\end{aligned}
$$

We will reject the two-sided null hypothesis H_0 when p-value, $2[1 - \Phi(|\mathsf{z}|)]$, is less than the significant level α.

DM-TKA Example

In DM-TKR Data, there are 5 infective patients of total 78 patients, the sample proportion is $6.41\% = 5/78$. The infectiive probability in U.S. is about 1%. Do our sample differ from U.S. population?

DM-TKA Example

1. We wish to test the null hypothesis and alternative hypothesis are

$$H_0 : \pi = \pi_0 (= 0.01) \quad \text{versus} \quad H_A : \pi \neq \pi_0.$$

2. We have collected the data.

3. The observed sample proportion ($\hat{\pi}$, test statistic) is 6.4%.

4. Let the significant level $\alpha = 0.05$, and $Z_{1-\alpha/2} = 1.960$.

DM-TKA Example

For two-sided test, the critical value (and critical region) for $\hat{\pi}$ is

$$\begin{aligned}
\pi_0 - Z_{1-\alpha/2}\frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{n}} &= 0.01 - 1.960 \times \frac{0.01 * 0.99}{\sqrt{78}} \\
&= 0.01 + 1.960 \times 0.01127 \\
&= -0.0121
\end{aligned}$$

and

$$\begin{aligned}
\pi_0 + Z_{1-\alpha/2}\frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{n}} &= 0.01 + 1.960 \times \frac{0.01 * 0.99}{\sqrt{78}} \\
&= 0.01 + 1.960 \times 0.01127 \\
&= 0.0322.
\end{aligned}$$

Critical values, $(c_{\alpha,1}, c_{\alpha,2})$, are $(-0.0121, 0.0322)$.

DM-TKA Example

We decide to reject the null hypothesis H_0 if

$$\hat{\pi} \;\; < \;\; -0.0121 = \pi_0 - Z_{1-\alpha/2}\frac{\sqrt{\pi(1-\pi)}}{\sqrt{n}}$$

or

$$\hat{\pi} \;\; > \;\; 0.0322 = \pi_0 + Z_{1-\alpha/2}\frac{\sqrt{\pi(1-\pi)}}{\sqrt{n}}.$$

Now the observed sample proportion $\hat{\pi} = 6.41\% > 0.0322$, so we reject the null hypothesis.

DM-TKA Example

1. The observed sample test statistic, z, is calculated as

$$z = \frac{(\hat{\pi} - \pi_0)}{\sqrt{\pi_0(1-\pi_0)}/\sqrt{n}} = \frac{(0.0641 - 0.01)}{\sqrt{0.01 \cdot 0.09}/\sqrt{78}} = 4.80.$$

2. The observed sample test statistic, z, is 4.80 which is greater than the Z critical value, $Z_{1-\alpha/2} = 1.960$.

3. So we reject the null hypothesis H_0.

DM-TKA Example

1. The two-sided $(1-\alpha)\times 100\%$ confidence interval for DM population proportion π based on the sample statistic, $\hat{\pi}$, can be calculated as

$$\begin{aligned} & \left(\hat{\pi} - Z_{1-\alpha/2} \times \frac{\sqrt{\hat{\pi}(1-\hat{\pi})}}{\sqrt{n}},\ \hat{\pi} + Z_{1-\alpha/2} \times \frac{\sqrt{\hat{\pi}(1-\hat{\pi})}}{\sqrt{n}}\right) \\ = & \ (0.00975, 0.1185). \end{aligned}$$

2. The $(1-\alpha)\times 100\%$ confidence interval for DM population proportion π is $(0.00975, 0.1185)$.

3. This interval does contain $\pi_0 = 0.01$.

4. So we do not reject the null hypothesis H_0.

DM-TKA Example

1. The p-value based on the observed sample test statistic, $\mathsf{z} = 4.80$, can be calculated as

$$2[1 - P(\hat{\pi} > |\bar{x}|) = 2[1 - \Phi(|\mathsf{z}|)] = \Phi(4.80) < 0.0001.$$

2. The p-value, < 0.0001, is less than the significant level $\alpha = 0.05$.

3. So we reject the null hypothesis.

DM-TKA Example: R

```
> y<-5; n<-78 # assign y and n in binomial
> alpha<-0.05 # assign significant level alpha
> pihat<-y/n  # sample proportion
> pihat
[1] 0.06410256
>  qihat<-1-pihat
>  se0<-sqrt(pi0*(1-pi0)/n) # s.e. under H0
>  se1<-sqrt(pihat*qihat/n) # s.e. Under HA
> Z1alpha<-qnorm(1-alpha/2) # Z_{1-alpha/2} quantile
> ztest<-(pihat-pi0)/se0    # sample Z test statistic
> ztest
[1] 4.802281
```

DM-TKA Example: R

```
> crit1<-pi0-Z1alpha*se0 # critical vale c_{alpha,1}
> crit2<-pi0+Z1alpha*se0 # critical value c_{alpha,2}
> crit1
[1] -0.01208098
> crit2
[1] 0.03208098
> Z.CI.L<-pihat-Z1alpha*se1 # C.I. Lower
> Z.CI.U<-pihat+Z1alpha*se1 # C.I. Upper
> Z.CI.L
[1] 0.009745922
> Z.CI.U
[1] 0.1184592
```

Power of One-sample Z Test for Proportion

1. The first thing is to decided the possible π value under H_A, since different π value under H_A will have different power.

2. Suppose in the DM-TKR example, we have $\pi_0 = 0.01$ and $\pi = \pi_A = 0.005$ or $\pi_A = 0.05$ under H_A.

3. What is power of the two-sided test?

$$\text{power} = P(\text{reject } H_0 \mid H_A \text{ is true})$$

4. Power is the probability of making the correct decision when the null hypothesis is not true. Specially,

$$\text{power} \;= 1 - \beta \;\;=\;\; P(\text{reject } H_0 : \pi = \pi_0 \mid H_A \text{ is true}).$$

Power

If $H_A : \pi = \pi_A < \pi_0$ is true, then we will reject H_0 when

$$\hat{\pi} < c_\alpha = \pi_0 - Z_{1-\alpha/2} \frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{n}} \tag{16}$$

or

$$\frac{(\hat{\pi} - \pi_A)}{\sqrt{\pi_A(1-\pi_A)}/\sqrt{n}} < \frac{(\pi_0 - Z_{1-\alpha/2}(\frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{n}}) - \pi_A)}{\sqrt{\pi_A(1-\pi_A)}/\sqrt{n}}. \tag{17}$$

Power

If $H_A : \pi = \pi_A < \pi_0$ is true,

$$\begin{aligned}
\text{power} &= 1-\beta = P(\hat{\pi} < c_\alpha \mid \pi = \pi_A) && (18)\\
&= P\Big(\hat{\pi} < \pi_0 - Z_{1-\alpha/2}\frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{n}} \mid \pi = \pi_A\Big) && (19)\\
&= P\Big(\mathsf{Z} = \frac{(\hat{\pi}-\pi_A)}{\sqrt{\pi_A(1-\pi_A)}/\sqrt{n}} < \frac{(\pi_0 - Z_{1-\alpha/2}(\frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{n}}) - \pi_A)}{\sqrt{\pi_A(1-\pi_A)}/\sqrt{n}} \mid \pi = \pi_A\Big)\\
&= P\Big(\mathsf{Z} < \frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{\pi_A(1-\pi_A)}}\Big(-Z_{1-\alpha/2} + \frac{(\pi_0-\pi_A)}{\sqrt{\pi_0(1-\pi_0)/n}}\Big) \mid \pi = \pi_A\Big)\\
&= \Phi\Big(\frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{\pi_A(1-\pi_A)}}\Big(-Z_{1-\alpha/2} + \frac{(|\pi_0-\pi_A|)}{\sqrt{\pi_0(1-\pi_0)/n}}\Big)\Big). && (20)
\end{aligned}$$

Power

If $H_A : \pi = \pi_A > \pi_0$ is true, then we will reject H_0 when

$$\hat{\pi} > c_\alpha = \pi_0 + Z_{1-\alpha/2}\frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{n}}, \tag{21}$$

or

$$\frac{(\hat{\pi} - \pi_A)}{\sqrt{\pi_A(1-\pi_A)}/\sqrt{n}} > \frac{(\pi_0 + Z_{1-\alpha/2}(\frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{n}}) - \pi_A)}{\sqrt{\pi_A(1-\pi_A)}/\sqrt{n}}. \tag{22}$$

Power

If $H_A : \pi = \pi_A > \pi_0$ is true,

$$\begin{aligned}
\text{power} &= 1-\beta = P(\hat{\pi} > c_\alpha \mid \pi = \pi_A) && (23)\\
&= P\Big(\hat{\pi} > \pi_0 + Z_{1-\alpha/2}\frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{n}} \mid \pi = \pi_A\Big) && (24)\\
&= P\Big(\mathsf{Z} = \frac{(\hat{\pi}-\pi_A)}{\sqrt{\pi_A(1-\pi_A)}/\sqrt{n}} > \frac{(\pi_0 + Z_{1-\alpha/2}(\frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{n}}) - \pi_A)}{\sqrt{\pi_A(1-\pi_A)}/\sqrt{n}} \mid \pi = \pi_A\Big)\\
&= P\Big(\mathsf{Z} > \frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{\pi_A(1-\pi_A)}}\Big(+Z_{1-\alpha/2} + \frac{(\pi_0-\pi_A)}{\sqrt{\pi_0(1-\pi_0)/n}}\Big) \mid \pi = \pi_A\Big)\\
&= 1-\Phi\Big(\frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{\pi_A(1-\pi_A)}}\Big(+Z_{1-\alpha/2} + \frac{(\pi_0-\pi_A)}{\sqrt{\pi_0(1-\pi_0)/n}}\Big)\Big)\\
&= \Phi\Big(\frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{\pi_A(1-\pi_A)}}\Big(-Z_{1-\alpha/2} + \frac{(|\pi_0-\pi_A|)}{\sqrt{\pi_0(1-\pi_0)/n}}\Big)\Big). && (25)
\end{aligned}$$

Power

For one-sided test, we use $Z_{1-\alpha}$ instead of $Z_{1-\alpha/2}$.

Power

So the power of the two-sided test $H_0 : \pi = \pi_0$ versus $H_A : \pi \neq \pi_0$ for the specific alternative $\pi = \pi_A$, where the underlying distribution is approximately normal and the population variance σ^2 is estimated as $\pi_A(1 - \pi_A)$, is given exactly by

$$\text{power} = \Phi\left[\frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{\pi_A(1-\pi_A)}}\left(-Z_{1-\alpha/2} + \frac{(|\pi_0 - \pi_A|)}{\sqrt{\pi_0(1-\pi_0)/n}}\right)\right]. \quad (26)$$

Power

1. The power formula has nothing to do with observed sample statistic z, however, the power depends on π_A and its variance $\pi_A(1 - \pi_A)$.

2. If we consider $H_A : \pi = \hat{\pi}$, that is, we calculated the power after the study, this is sometime called **post-hoc power**. It is not recommend by many statisticians.

3. For one-sided test, we use $Z_{1-\alpha}$ (instead of $Z_{1-\alpha/2}$).

DM-TKA Example

1. Suppose $H_A : \pi = \pi_A$, for example, $\pi_A = 0.05$. We have power

$$\Phi\Big(\frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{\pi_A(1-\pi_A)}}\Big(\frac{(|\pi_0 - \pi_A|)}{\sqrt{\pi_0(1-\pi_0)/n}}\Big)\Big) = 0.76612.$$

2. If we assume $H_A : \pi_A = 0.09$, then the power is

$$\Phi\Big(\frac{\sqrt{\pi_0(1-\pi_0)}}{\sqrt{\pi_A(1-\pi_A)}}\Big(-Z_{1-\alpha/2} + \frac{(|\pi_0 - \pi_A|)}{\sqrt{\pi_0(1-\pi_0)/n}}\Big)\Big) = 0.9630.$$

DM-TKA Example

The power depends on the variance of π_A, so the direction of power of the approximate Z test for proportion is not as clear as normal distribution.

Note: the power formula has nothing to do with observed sample statistic z. If we consider $H_A : \pi = \hat{\pi}$, that is, we calculated the power after the study, this is sometime called **post-hoc power**. It is not recommend by many statisticians.

DM-TKA Example: R

```
>  # two-tailed power
> power.prop.two.sided.Z.test<-function(pi0,pia,alpha,n)
    { power<-sqrt((pi0*(1-pi0))/(pia*(1-pia)) )
        *(-qnorm(1-alpha/2)
        +abs(pi0-pia)*sqrt(n)/sqrt(pi0*(1-pi0)))
    power<-pnorm(power)
    cat("power = ",power,"\n")
  }
> power.prop.two.sided.Z.test(0.01,0.05,0.05,78)
power =  0.7661206
> power.prop.two.sided.Z.test(0.01,0.09,0.05,78)
power =  0.9630652
```

Exact Small-Sample Inference

Exact Test for Proportion

Exact Test for Proportion

1. The approximate Z test procedure to test the hypothesis $H_0 : \pi = \pi_0$ versus $H_A : \pi \neq \pi_0$ depends on the assumptions is only true if $n\pi_0(1 - \pi_0) \geq 5$.

2. With modern computational power, it is not necessary to rely on large-sample approximation for the distribution of statistics such as $\hat{\pi}$.

3. Tests and confidence intervals can use the binomial distribution directly rather than its normal approximation. Such inferences occur naturally for small samples, but apply for any n.

Exact Test for Proportion

1. We will base our test on **exact** binomial probabilities.

2. Let $Y \sim Bin(n, \pi_0)$ under H_0.

3. Let $\hat{\pi} = y/n$, be the observed sample proportion.

4. The computation of the p-value depends on whether $\hat{\pi} \leq \pi_0$ or $\hat{\pi} \geq \pi_0$.

p-value of the Exact Test for Proportion

1. If $\hat{\pi} \leq \pi_0$, then

$$p-\text{value}/2 = P(Y \leq y \text{ successes in } n \text{ trials} \mid H_0) \tag{27}$$

$$= \sum_{k=0}^{y} \binom{n}{k} \pi_0^k (1-\pi_0)^{n-k} \tag{28}$$

2. If $\hat{\pi} \geq \pi_0$, then

$$p-\text{value}/2 = P(Y \geq y \text{ successes in } n \text{ trials} \mid H_0) \tag{29}$$

$$= \sum_{k=y}^{n} \binom{n}{k} \pi_0^k (1-\pi_0)^{n-k} \tag{30}$$

p-value of the Exact Test for Proportion

1. If $\hat{\pi} \leq \pi_0$, then

$$\begin{aligned} p - \text{value} &= 2 \times P(Y \leq y) && (31) \\ &= \min\left[2 \sum_{k=0}^{y} \binom{n}{k} \pi_0^k (1-\pi_0)^{n-k}, \quad 1\right] && (32) \end{aligned}$$

2. If $\hat{\pi} \geq \pi_0$, then

$$\begin{aligned} p - \text{value} &= 2 \times P(Y \geq y) && (33) \\ &= \min\left[2 \sum_{k=y}^{n} \binom{n}{k} \pi_0^k (1-\pi_0)^{n-k}, \quad 1\right] && (34) \end{aligned}$$

p-value of the Exact Test for Proportion

We illustrate by testing $H_0 : \pi = 0.5$ against $H_A : \pi \neq 0.5$ for the survey results, $y = 0$, with $n = 25$. We noted that the score statistic equals $z = -5.0$. The exact p-value for this statistic, based on the null $Bin(25, 0.5)$ distribution, is

$$P(\,|\,z\,|\, \geq 5.0) = P(Y = 0 \text{ or } Y = 25) = 0.5^{25} + 0.5^{25} = 0.00000006.$$

C.I. of the Exact Test for Proportion

1. The exact $100(1-\alpha)\%$ confidence intervals consists of all π for which p-values exceed α in exact binomial tests.

2. The best known interval (**Clopper and Person**, 1934) uses the tail method for forming confidence intervals.

3. It requires each one-sided p-value to exceed $\alpha/2$.

C.I. of the Exact Test for Proportion

The lower and upper endpoints are the solutions in π_0 to the equations

$$\sum_{k=y}^{n} \binom{n}{k} \pi_0^k (1-\pi_0)^{n-k} = \alpha/2$$

$$\text{and} \qquad \sum_{k=0}^{y} \binom{n}{k} \pi_0^k (1-\pi_0)^{n-k} = \alpha/2, \tag{35}$$

except that the lower bound is 0 when $y = 0$ and the upper bound is 1 when $y = n$.

C.I. of the Exact Test for Proportion

1. The **Clopper and Person** confidence interval equals

$$\left[1+\frac{n-y+1}{y\ F_{2y,2(n-y+1),\alpha/2}}\right]^{-1} < \ \pi \ < \ \left[1+\frac{n-y+1}{(y+1)\ F_{2(y+1),2(n-y),(1-\alpha/2)}}\right]^{-1}, \qquad (36)$$

   where $F_{a,b,c}$ denotes the c quantile form the F distribution with degrees of freedom a and b.

2. Example,

   When $y = 0$ with $n = 25$, the **Clopper-Pearson** 95% **confidence interval** for π is $(0, 0.137)$.

DM-TKA Example: Exact Test

The exact $(1-\alpha)\times 100\%$ confident interval is $(0.0211, 0.1433)$. The exact two-sided test p-value is $0.001152 < \alpha = 0.05$ (SAS: $p = 0.0023$). We reject the null hypothesis H_0 based on the exact confidence interval and p-value.

DM-TKA Example: Exact Test with R binom.test

```
> binom.test(x=5,n=78,p=0.01,alternative = c("two.sided"),
    conf.level = 0.95)
        Exact binomial test
data:  5 and 78
number of successes=5, number of trials=78, p-value=0.001152
alternative hypothesis:
true probability of success is not equal to 0.01
95 percent confidence interval:
 0.02113972 0.14328760
sample estimates:
probability of success
            0.06410256
```

DM-TKA Example: Exact C.I. and Asymptotic C.I. with R

```
> library(Hmisc)
> help(binconf)
> binconf(x=5,n=78,alpha=0.05,method="all")
             PointEst       Lower     Upper
Exact      0.06410256 0.021139720 0.1432876
Wilson     0.06410256 0.027689315 0.1414360
Asymptotic 0.06410256 0.009745922 0.1184592
```

DM-TKA Example: R

```
> help(prop.test)
> prop.test(x=5,n=78,p=0.01,alternative = c("two.sided"),
    correct=F,conf.level = 0.95)
    1-sample proportions test without continuity correction
data:  5 out of 78, null probability 0.01
X-squared = 23.0619, df = 1, p-value = 1.569e-06
alternative hypothesis: true p is not equal to 0.01
95 percent confidence interval:
 0.02768932 0.14143595
sample estimates:
         p
0.06410256
```

DM-TKA Example: Exact Test with R

```
Warning message:
Chi-squared approximation may be incorrect in:
prop.test(x=5, n=78, p=0.01,alternative=c("two.sided"),
```

DM-TKA Example: SAS

```
title "FREQ: One-sample Z test
       for proportion with 95% C.I." ;
proc freq data=dmtakcsv order=data ;
    exact binomial ;
    tables infect / bin( p=0.01) ;
run;
```

DM-TKA Example: SAS

```
title "Categroical Data:
    Graphics of One-sample";
proc gchart data=dmtakcsv ;
    vbar infect / discrete ;
    hbar infect / discrete ;
    pie infect / discrete ;
    pie infect / discrete
     explode=1 slice=arrow percent=inside ;
run;
```

DM-TKA Example: SAS

```
The FREQ Procedure
                                         Cumulative    Cumulative
infect    Frequency      Percent          Frequency       Percent
-------------------------------------------------------------------
     1            5         6.41                  5          6.41
     0           73        93.59                 78        100.00
```

DM-TKA Example: SAS

```
Binomial Proportion for infect = 1
-----------------------------------
Proportion (P)                0.0641
ASE                           0.0277
95% Lower Conf Limit          0.0097
95% Upper Conf Limit          0.1185
```

DM-TKA Example: SAS

```
Exact Conf Limits
95% Lower Conf Limit          0.0211
95% Upper Conf Limit          0.1433
```

DM-TKA Example: SAS

```
Test of H0: Proportion = 0.01

ASE under H0                  0.0113
Z                             4.8023
One-sided Pr >  Z             <.0001
Two-sided Pr > |Z|            <.0001
```

DM-TKA Example: SAS

```
Exact Test

One-sided Pr >=  P              0.0012

Two-sided = 2 * One-sided       0.0023


Sample Size = 78
```