

Two-Sample Test for Proportion

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Approximated Two-Sample Z Test for Proportion

DM-TKA Example

In DM-TKR Data, the infection In DM-TKR-DATA, the infection proportion for non-adding antibiotics group is 13.5%(5/37) and 0%(0/41) for adding antibiotics group. Is there any different between two groups?

Notations for Two-Sample Proportions

Table 1: Notation for two sub-population proportion

Quantity	Sub-population	
	1	2
Sample size	n_1	n_2
Population proportion “success”	π_1	π_2
Population variance of proportion	$\pi_1(1 - \pi_1)$	$\pi_2(1 - \pi_2)$
Sample number of “success”	Y_1	Y_2
Sample proportion	$\hat{\pi}_1$	$\hat{\pi}_2$
Sample variance of estimated sample proportion	$\hat{\pi}_1(1 - \hat{\pi}_1) / \sqrt{n_1}$	$\hat{\pi}_2(1 - \hat{\pi}_2) / \sqrt{n_2}$
Observed single sample proportion	$\hat{\pi}_1$	$\hat{\pi}_2$
Observed variance of estimated sample proportion	$\hat{\pi}_1(1 - \hat{\pi}_1) / \sqrt{n_1}$	$\hat{\pi}_2(1 - \hat{\pi}_2) / \sqrt{n_2}$

Hypothesis

Table 2: Three types of null hypothesis and alternative hypothesis

H_0 Null Hypothesis	H_A Different means	H_A Difference between means
$H_0 : \pi_1 = \pi_2$	$H_A : \pi_1 > \pi_2$	$H_A : \pi_1 - \pi_2 > 0$
$H_0 : \pi_1 = \pi_2$	$H_A : \pi_1 < \pi_2$	$H_A : \pi_1 - \pi_2 < 0$
$H_0 : \pi_1 = \pi_2$	$H_A : \pi_1 \neq \pi_2$	$H_A : \pi_1 - \pi_2 \neq 0$

Sample Difference and Test Statistic for Proportion

With central-limit theorem

$$H_0 : \pi_1 - \pi_2 = 0 \quad (1)$$

$$\text{Estimator of difference: } \hat{\pi}_1 - \hat{\pi}_2 \quad (2)$$

$$\text{Estimator of common proportion: } \hat{\pi} = \frac{Y_1 + Y_2}{n_1 + n_2} \quad (3)$$

Sample Difference and Test Statistic for Proportion

Standard error of sample difference of proportion Under H_A :

$$\sigma_{\pi_1 - \pi_2} = \sqrt{\frac{\pi_1(1 - \pi_1)}{n_1} + \frac{\pi_2(1 - \pi_2)}{n_2}} \quad (4)$$

Estimated sample standard error under H_0 :

$$SE^0_{\hat{\pi}_1 - \hat{\pi}_2} = \sqrt{\frac{\hat{\pi}_1(1 - \hat{\pi}_1)}{n_1} + \frac{\hat{\pi}_2(1 - \hat{\pi}_2)}{n_2}} \quad (5)$$

$$= \sqrt{\hat{\pi}(1 - \hat{\pi}) \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}, \quad (6)$$

Sample Difference and Score-Based Test Statistic for Proportions

Score-based Z test statistic:
$$Z = \frac{(\hat{\pi}_1 - \hat{\pi}_2)}{\sqrt{\hat{\pi}(1 - \hat{\pi}) \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}} \quad (7)$$

Observed test statistic:
$$z = \frac{(\hat{\pi}_1 - \hat{\pi}_2)}{\sqrt{\hat{\pi}(1 - \hat{\pi}) \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}} \quad (8)$$

Approximated two-sided p -value $= 2 \times [1 - \Phi(|z|)] \quad (9)$

Critical Regions

The three different critical regions for rejection of the three different types of alternative hypothesis, H_A , for Z test are in Table (3).

Table 3: Three critical regions for rejection of 3 types of alternative hypothesis, H_A

Test for alternative hypothesis	rejection $H_0 : \pi_1 = \pi_2$
$H_A : \pi_1 > \pi_2$	$z > Z_{1-\alpha}$
$H_A : \pi_1 < \pi_2$	$z < Z_\alpha$
$H_A : \pi_1 \neq \pi_2$	$ z > Z_{1-\alpha/2}$

$z_{1-\alpha}$ is the percentile of standard normal distribution such that $1 - \Phi(Z_{1-\alpha}) = \alpha$.

Approximate Z Test Statistic with Binomial Correction

Z test statistic with binomial correction:

$$Z^{\star} = \frac{|\hat{\pi}_1 - \hat{\pi}_2| - \frac{1}{2}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}{\sqrt{\hat{\pi}(1 - \hat{\pi})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \quad (10)$$

Wald-Based Approximate Z Test Statistic

When the underlying proportions π_1 and π_2 are not hypothesized to be equal, a good estimate of the standard error of $\pi_1 - \pi_2$ is

Standard error under H_A :

$$\sigma_{\pi_1 - \pi_2} = \sqrt{\frac{\pi_1(1 - \pi_1)}{n_1} + \frac{\pi_2(1 - \pi_2)}{n_2}} \quad (11)$$

Observed sampled standard error under H_A :

$$SE_{\hat{\pi}_1 - \hat{\pi}_2} = \sqrt{\frac{\hat{\pi}_1(1 - \hat{\pi}_1)}{n_1} + \frac{\hat{\pi}_2(1 - \hat{\pi}_2)}{n_2}} \quad (12)$$

Wald-Based Approximate Z Test Statistic

The new feature of the standard error is based on a pooled estimate of the proportion. When the null hypothesis H_0 is true, the common variance of sample proportion $\hat{\pi}(1 - \hat{\pi})$ is better.

Sample Wald-based Z test statistic:

$$Z^* = \frac{|\hat{\pi}_1 - \hat{\pi}_2|}{\sqrt{\frac{\hat{\pi}_1(1-\hat{\pi}_1)}{n_1} + \frac{\hat{\pi}_2(1-\hat{\pi}_2)}{n_2}}} \quad (13)$$

Sample score-based Z test statistic (better):

$$Z = \frac{|\hat{\pi}_1 - \hat{\pi}_2|}{\sqrt{\hat{\pi}(1 - \hat{\pi}) \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}} \quad (14)$$

Wald-Based Confidence Interval

The $(1 - \alpha) \times 100\%$ confidence interval (under H_A) based on the observed sample is calculated as

$$\left(\hat{\pi}_1 - \hat{\pi}_2 - Z_{1-\alpha/2} \times \sqrt{\frac{\hat{\pi}_1(1 - \hat{\pi}_1)}{n_1} + \frac{\hat{\pi}_2(1 - \hat{\pi}_2)}{n_2}}, \right. \\ \left. \hat{\pi}_1 - \hat{\pi}_2 + Z_{1-\alpha/2} \times \sqrt{\frac{\hat{\pi}_1(1 - \hat{\pi}_1)}{n_1} + \frac{\hat{\pi}_2(1 - \hat{\pi}_2)}{n_2}} \right) \quad (15)$$

Wald-Based Confidence Interval

1. When the $(1 - \alpha) \times 100\%$ confidence interval does not include 0, then we also reject H_0 to test the alternative $H_A : \pi_1 \neq \pi_2$ at α significant level.
2. However, the consistency just found between the test for the significance of the difference between π_1 and π_2 and the confidence intervals for $\pi_1 - \pi_2$ will not obtain.
3. For example, the Z test (7) may fail to reject the hypothesis that $H_0 : \pi_1 = \pi_2$, but the confidence interval (15) may exclude the value zero, see Eberhardt and Flinger (1997), Robbins (1997).

Z-Test and Chi-Square Test for Proportion

Note: The large sample Z -test for two proportions is nearly equivalent to the commonly used chi-square test, (chi-square is nearly equal to the square of the Z -statistic). The chi-square test is not described in this chapter.

DM-TKA Example

1. In DM-TKR-DATA, the infection proportion for non-adding antibiotics group is 13.5%(5/37) and 0%(0/41) for adding antibiotics group.
2. The sample estimated difference of proportion is 13.5% and the approximated 95% confidence of difference is (2.5%, 24.5%).
3. The Z test is $0.1351/0.0562 = 2.40$ ($Z^2 = 5.77$), p -value is 0.016.
4. We concluded that the infection proportion is significant different between nonadding antibiotics and adding antibiotics group.

DM-TKA Example: Note

The sample infective proportion of the adding-antibiotic group is 0, so different software have different calculation methods such that the results would not similar.

DM-TKA Example: R

```
> setwd("C://temp//Rdata")
> DMTKRcsv<-read.csv("DMTKRcsv.csv",
  header = TRUE, sep = ",", dec=".")
> DMTKRcsv
> attach(DMTKRcsv)
> attach(DMTKRcsv)
>
> table(ABS,INFECT)
```

	INFECT	
ABS	0	1
0	32	5
1	41	0

DM-TKA Example: R

```
> prop.test(c(5,0),c(36,41),correct=F)
2-sample test for equality of proportions
without continuity correction
data:  c(5, 0) out of c(36, 41)
X-squared = 6.0899, df = 1, p-value = 0.01360
alternative hypothesis: two.sided
95 percent confidence interval:
 0.02591965 0.25185812
sample estimates:
   prop 1    prop 2 
0.1388889 0.0000000
```

DM-TKA Example: R

Warning message:

Chi-squared approximation may be incorrect in:

```
prop.test(c(5, 0), c(36, 41), correct = F)
```

DM-TKA Example: SAS

DM-TKA Example: SAS

```
Title1 "FREQ: Two-sample Z test  
        for proportion wiht 95% C.I.";  
proc freq data=dmtkrcsv order=data;  
    tables abs*infect / riskdiff ;  
run;
```

DM-TKA Example: SAS

FREQ: Two-sample Z test for proportion wiht 95% C.I.

The FREQ Procedure

Statistics for Table of abs by infect

		Column 1 Risk Estimates				
		(Asymptotic) 95%			(Exact) 95%	
	Risk	ASE	Confidence Limits		Confidence Limits	

Row 1	0.1351	0.0562	0.0250	0.2453	0.0454	0.2877
Row 2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0860
Total	0.0641	0.0277	0.0097	0.1185	0.0211	0.1433
Difference	0.1351	0.0562	0.0250	0.2453		
Difference is (Row 1 - Row 2)						

DM-TKA Example: SAS

Difference is (Row 1 - Row 2)

Column 2 Risk Estimates

			(Asymptotic) 95%		(Exact) 95%	
	Risk	ASE	Confidence Limits		Confidence Limits	

Row 1	0.8649	0.0562	0.7547	0.9750	0.7123	0.9546
Row 2	1.0000	0.0000	1.0000	1.0000	0.9140	1.0000
Total	0.9359	0.0277	0.8815	0.9903	0.8567	0.9789
Difference	-0.1351	0.0562	-0.2453	-0.0250		
Difference is (Row 1 - Row 2)						

Sample Size = 78