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Notes on Black-Scholes Option Pricing Formula

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These notes are a brief introduction to the Black-Scholes formula, which prices the European call options. The essential reading is of course their 1973 *Journal of Political Economy* paper. Other standard references are Darrell Duffie's *Dynamic Asset Pricing Theory*, 3rd ed., 2001, and John Campbell, Andrew Lo, and Craig MacKinlay's *The Econometrics of Financial Markets*, 1997, both published by the Princeton University Press. Here I follow Duffie's text more closely.

The basic tools to understand continuous-time finance are Brownian motion (or Wiener process). A recent book written (not yet published) by Professor Nancy Stokey at the University of Chicago is an excellent introduction to these tools and their applications. Readers could visit her website (http://home.uchicago.edu/∼ntokey/course.htm) for details.

1. Basics of Brownian Motion

There are two basic types of Brownian motion: arithmetic and geometric. The latter was used in option pricing. The first type is defined as follows:

Definition

An arithmetic Brownian motion B_t is a stochastic process which satisfies:

(a) $B_s - B_t \sim N(\mu(s-t), \sigma^2(s-t)), \forall s > t$,

(b) $\forall t_i, 0 \leq t_0 < t_1 < ... < \infty, B(t_0), B(t_1) - B(t_0), B(t_2) - B(t_1), ...$, are independently distributed,

(c) sample path of B_t is continuous,

(d) $B_0 = 0$, a.s. (almost surely).

Definition

A standard Brownian motion is an arithmetic Brownian motion when $\mu = 0$ and $\sigma = 1$. Usually we call μ *drift*, and σ *diffusion process*, respectively.

Note: Since Brownian motion is a continuous-time random walk, $B(t + \Delta t) - B(t)$ can be viewed as Gaussian white noise as $\Delta t \to 0$, or "$dB(t)$" is a "continuous-time version" of white noise.

Example

Let S_t be stock price at t, and assume that it is an arithmetic Brownian motion. Let B_t be a standard Brownian motion. Then we could write the stock price as:

(1) $S_t = \mu t + \sigma B_t$

which satisfies all the requirements of conditions (a)-(d) above.

2. Stochastic Calculus and Ito's Lemma

Now we can write a *stochastic differential equation*:

(2) $dS_t = \mu dt + \sigma dB_t$

which has a *stochastic integral* counterpart as follows:

(3) $S_t = x + \int_0^t \mu ds + \int_0^t \sigma dB_s$

where $S_0 = x$ is an integral constant.

Definition

A stochastic process with a representation of equation (3) is called an *Ito process.*

Ito's Lemma:

For an Ito process $dX_t = \mu_t dt + \sigma_t dB_t$, $Y_t = f(X_t, t)$ ($f : R^2 \to R$ is twice-differentiable) is also an Ito process which satisfies:

(4) $dY_t = [f_x(X_t, t)\mu_t + f_t(X_t, t) + \frac{1}{2} f_{xx}(X_t, t)\sigma_t^2]dt + f_x(X_t, t)\sigma_t dB_t$

Definition:

S_t is a geometric Brownian motion (or log-normal), if

(5) $S_t = x\exp(\alpha t + \sigma B_t)$,

$\forall t \geq 0$, $x > 0$, where α, σ are constants. Since $\alpha t + \sigma B_t = \int_0^t \alpha ds + \int_0^t \sigma dB_s$ is an Ito process, Ito's lemma implies that S_t is an Ito process and $\ln S_t$ is also an Ito process (note that both exponential and logarithmic functions are twice-differentiable), so we have

(6) $d\ln S_t = \mu dt + \sigma dB_t$, $S_0 = x$, $\mu = \alpha + \frac{\sigma^2}{2}$, or

(6)' $dS_t = \mu S_t dt + \sigma S_t dB_t$

3. Black-Scholes Formula

Now we turn to the derivation of Black-Scholes formula. The basic idea behind this formula is an arbitrage equilibrium among three assets: stock, bond, and European call option. It is a *risk-neutral valuation* because investors in their model economy were implicitly assumed to be risk neutral and they are concerned only with maximizing profits. A more realistic, discrete-time formulation of option pricing (by using a *binomial tree*) was later proposed by William Sharpe, and formalized by Cox-Ross-Rubinstein (*Journal of Financial Economics*, 1979), which would not be discussed here.

A bond price is defined by $\beta_t = \beta_0 e^{rt}$, where r is the continuously compounding interest rate. Taking logarithm on both sides and rearranging terms, we would have

(7) $d\beta_t = r\beta_t dt$

Definition (*Self-financing condition*): for trading strategies (a, b),

$a_t S_t + b_t \beta_t = a_0 S_0 + b_0 \beta_0 + \int_0^t a_\tau dS_\tau + \int_0^t b_\tau d\beta_\tau$

This means that current portfolio value (the left-hand-side of the above equation) is equal to the sum of initial investment (the first two terms in the right-hand-side) and trading gains (the last two terms of right-hand-side). And trading gains are the gains from trading stocks and bonds. These two are *primary* or *underlying assets*, and the *derivative asset* in the present case would be the European call options. The pricing formula of the associated *American call options* was derived in the same year (1973) by Robert C. Merton in a paper published in the *Bell Journal of Economics*. The corresponding pricing formula for *put options* can be obtained by the *put-call parity*, which will be briefly discussed at the end of these notes.

Consider an European call option, the price of which at the expiry date Y_T being

(8) $Y_T = \max(S_T - K, 0) \equiv (S_T - K)^+$

where K is the exercise or striking price of the call option.

No Arbitrage Condition:

$Y_t = a_t S_t + b_t \beta_t$

This condition requires that in equilibrium risk-neutral investors be indifferent between holding underlying assets (with a_t stocks and b_t bonds) and holding derivative asset (European call options). Investors could not increase their profits by either changing the position of stocks/bonds or changing the position of call options. So no arbitrage is the theoretical basis of Black-Scholes formula.

Now we are ready to derive the Black-Scholes formula. Let $Y_t = C(S_t, t)$ be the value of European call option at period t, then by Ito's lemma,

(9) $dY_t = [C_x(S_t,t)\mu S_t + C_t(S_t,t) + \frac{1}{2}C_{xx}(S_t,t)\sigma^2 S_t^2]dt + C_x(S_t,t)\sigma S_t dB_t$

By no arbitrage condition and the properties of stochastic differential equations, we have

(10) $dY_t = a_t dS_t + b_t d\beta_t = a_t(\mu S_t dt + \sigma S_t dB_t) + b_t(r\beta_t dt)$
$\quad = (a_t \mu S_t + b_t r \beta_t)dt + a_t \sigma S_t dB_t$

By the *method of undetermined coefficients*, (9), (10) imply

(11-1) $C_x(S_t,t)\mu S_t + C_t(S_t,t) + \frac{1}{2}C_{xx}(S_t,t)\sigma^2 S_t^2 = a_t \mu S_t + b_t r \beta_t$

(11-2) $C_x(S_t,t)\sigma S_t = a_t \sigma S_t$ (and this implies $a_t = C_x(S_t,t)$)

Because $a_t = C_x(S_t,t)$, by no arbitrage condition and $Y_t = C(S_t,t)$, we have

$C_x(S_t,t)S_t + b_t\beta_t = C(S_t,t)$.

$b_t = [C(S_t,t) - C_x(S_t,t)S_t]/\beta_t$.

By (11-1),

$C_t(S_t,t) + \frac{1}{2}C_{xx}(S_t,t)\sigma^2 S_t^2 = b_t r \beta_t = r[C(S_t,t) - C_x(S_t,t)S_t]$,

and therefore

$-rC(S_t,t) + C_t(S_t,t) + rS_t C_x(S_t,t) + \frac{1}{2}C_{xx}(S_t,t)\sigma^2 S_t^2 = 0$

If $x = S_t$ is the current stock price, and at the expiry date $Y_T = C(S_T,T) = (S_T - K)^+$, then there is a *boundary condition*:

(12) $C(x,T) = (x-K)^+$

$x \in (0,\infty)$, with the following partial differential equation:

(13) $-rC(x,t) + C_t(x,t) + rxC_x(x,t) + \frac{1}{2}\sigma^2 x^2 C_{xx}(x,t) = 0$

Black-Scholes Formula:

(14) $C(x,t) = x\phi(z) - e^{-r(T-t)}K\phi(z - \sigma\sqrt{T-t})$,

where $z = [\ln(x/K) + (r + \frac{\sigma^2}{2})(T-t)]/(\sigma\sqrt{T-t})$, and ϕ is the cumulative standard normal distribution function. Mathematically, this formula is a solution to equations (12) and (13).

4. Put-Call Parity

Finally we want to discuss how to derive the corresponding pricing formula for the European put options. There is a relationship between call and put options, which naturally arises because of the arbitrage conditions in financial markets equilibrium.

Put-Call Parity:

(15) $S_t + P_t - C_t = e^{-r(T-t)}K$

$\forall t$, $t = 0, 1, 2, ..., T$. P_t is the value of put option at period t.

An intuition behind this relation is that this is an arbitrage equilibrium between using options (for both put and call ones) and not using; or between "in (or out of) the money" and "at the money".

More precisely, the benefits when using options are as follows. If the expected end-of-period (at the expiry date) stock price is larger (smaller) than the striking price, that is if $S_T > (<)K$, then we call this situation *in the money* for the call option or *out of the money* for the put option (in the money for the put option or out of the money for the call option), and the investor should sell (buy) the call (put) option to maximize her profit. Otherwise, the investor should do nothing but take as given $S_T = K$ at the expiry date T, and its present value from the period t on would be just $e^{-r(T-t)}K$. We call this situation *at the money*. When capital can be freely moved between these two margins, equation (15) would hold in arbitrage equilibrium.

A *sloppy* illustration of this parity is as follows. If at period t, the expected $S_T > K$, then this means that call option is in the money and put option is out of the money, or equivalently $C_T > P_T$. On the other hand, if at period t, the expected $S_T < K$, then this means that put option is in the money and call option is out of the money, or equivalently $C_T < P_T$. And finally if at period t, the expected $S_T = K$, then this means that both call option and put option are at the money, or equivalently $C_T = P_T$. In the first situation the investor should sell the call option to increase her profit. In the second situation the investor should buy the put option, and in the last one she should do nothing. Put all these stuffs together we would *guess* that in arbitrage equilibrium the relationship among these variables could be described exactly by equation (15).