The Appendix of the paper "Should the Landlord Port Authority Negotiate Concession Contracts with Terminal Operators?"

<u>Proof of Lemma 1</u>: Given that the proofs mirror those presented by Liu et al. (2018), they have been omitted.

<u>Proof of Proposition 1</u>:

Under $c_1 = c_2 = c$ and $\underline{U}_1 = \underline{U}_2 \equiv \underline{U}$, we have $\delta_1 = \delta_2 = \frac{a-r-c}{2+b} > 0$, and equilibrium cargo-handling amounts of operators 1 and 2 are $q_1^T = q_2^T = \frac{a-r-c}{2+b}$ for $\delta \in [0, \delta_1]$ and $q_1^T = q_2^T = \delta$ for $\delta \in (\delta_1, \frac{a}{1+b})$ by Lemma 1. Consequently, we can deduce the solutions to the problem outlined in (7) based on the following two scenarios. In both instances, our attention will be directed towards cases where $\pi^T > \underline{U}$ and $R^T > \underline{R}$ hold, as $W = 0$ implied by $\pi^T = \underline{U}$ or $R^T = \underline{R}$ is not deemed noteworthy.

<u>Case 1</u>: Suppose $\delta \in [0, \delta_1]$. We then have $q_1^T = q_2^T \equiv q^T = \dfrac{a-r-c}{2+b}$, $\pi_1^T = \pi_2^T = \left(q^T\right)^2 - f \equiv \pi^T$ and $R = 2rq^T + 2f$ by Lemma 1. Accordingly, the problem in (7) becomes

$$\max_{r\geq 0, f\geq 0, \delta\geq 0} W \equiv [R - \underline{R}]^{(1-\beta_1-\beta_2)}[\pi^T - \underline{U}]^{\beta_1}[\pi^T - \underline{U}]^{\beta_2}$$

$$\text{s.t. } 0 \leq \delta \leq \delta_1,\ 0 \leq r < \bar{r},\ f \geq 0,\ \pi^T \geq \underline{U} \text{ and } r(q_1^T + q_2^T) + 2f \geq \underline{R}. \tag{A1}$$

The Lagrange function associated with the problem in (A1) is

$$L = \begin{Bmatrix} \left[2rq^T + 2f - \underline{R}\right]^{(1-\beta_1-\beta_2)}\left[\pi^T - \underline{U}\right]^{\beta_1}\left[\pi^T - \underline{U}\right]^{\beta_2} + \lambda_1\left(\delta_1 - \delta\right) \\ +\lambda_2\left(\bar{r} - r\right) + \lambda_3\left(\pi^T - \underline{U}\right) + \lambda_4\left[2rq^T + 2f - \underline{R}\right] \end{Bmatrix},$$

where λ_1, λ_2, λ_3 and λ_4 are respective Lagrange multipliers associated with the five inequality constraints in (A1) except for $r \geq 0$, $f \geq 0$ and $\delta \geq 0$. The corresponding Kuhn-Tucker conditions are as follows.

$$\frac{\partial L}{\partial r} = \begin{Bmatrix} D \cdot \left\{(1-\beta_1-\beta_2)\frac{\partial R}{\partial r}\left[\pi^T - \underline{U}\right]^2 - \frac{2q^T}{2+b}\left[R - \underline{R}\right]\left[\pi^T - \underline{U}\right]\left(\beta_1 + \beta_2\right)\right\} \\ -\frac{1}{2+b}\lambda_1 - \lambda_2 - \frac{2q^T}{2+b}\lambda_3 + \frac{2\left(a-2r-c\right)}{2+b}\lambda_4 \end{Bmatrix} \leq 0,\ r \cdot \frac{\partial L}{\partial r} = 0, \tag{A2}$$

$$\frac{\partial L}{\partial f} = D \cdot \left\{2(1-\beta_1-\beta_2)\left[\pi^T - \underline{U}\right]^2 - \left[R - \underline{R}\right]\left[\pi^T - \underline{U}\right]\left(\beta_1 + \beta_2\right)\right\} - \lambda_3 + 2\lambda_4 \leq 0,\ f \cdot \frac{\partial L}{\partial f} = 0, \tag{A3}$$

$$\frac{\partial L}{\partial \delta} = -\lambda_1 \leq 0,\ \delta \cdot \frac{\partial L}{\partial \delta} = 0, \tag{A4}$$

$$\frac{\partial L}{\partial \lambda_1} = \frac{a-r-c}{2+b} - \delta \geq 0,\ \lambda_1 \cdot \frac{\partial L}{\partial \lambda_1} = 0, \tag{A5}$$

$$\frac{\partial L}{\partial \lambda_2} = \overline{r} - r \geq 0,\ \lambda_2 \cdot \frac{\partial L}{\partial \lambda_2} = 0. \tag{A6}$$

$$\frac{\partial L}{\partial \lambda_3} = \pi^T - \underline{U} \geq 0,\ \lambda_3 \cdot \frac{\partial L}{\partial \lambda_3} = 0 \text{ and} \tag{A7}$$

$$\frac{\partial L}{\partial \lambda_4} = r(q_1^T + q_2^T) + 2f - \underline{R} \geq 0,\ \lambda_4 \cdot \frac{\partial L}{\partial \lambda_4} = 0, \tag{A8}$$

where $D = \left\{[R-\underline{R}]^{-(\beta_1+\beta_2)}[\pi_1^T - \underline{U}_1]^{-(1-\beta_1)}[\pi_2^T - \underline{U}_2]^{-(1-\beta_2)}\right\} > 0$.

Since we focus on the cases of $\pi^T > \underline{U}$ and $R > \underline{R}$, we have $\lambda_3^* = \lambda_4^* = 0$ by (A7)-(A8) and $\lambda_2^* = 0$ in (A6) by $r < \overline{r}$. Conversely, if $\lambda_1 > 0$, then (A5) implies $\delta^T = \delta_1 > 0$, and thus $\lambda_1^* = 0$ by (A4), resulting in a contradiction. Hence, we conclude $\lambda_1^* = 0$. Consequently, the conditions in (A2)-(A3) are transformed into

$$\frac{\partial L}{\partial r} = D \cdot \left\{(1-\beta_1-\beta_2)\frac{\partial R}{\partial r}\left[\pi^T - \underline{U}\right]^2 - \frac{2q^T}{2+b}[R-\underline{R}]\left[\pi^T - \underline{U}\right](\beta_1+\beta_2)\right\} \leq 0,\ r \cdot \frac{\partial L}{\partial r} = 0 \text{ and}$$

$$\frac{\partial L}{\partial f} = D \cdot \left\{2(1-\beta_1-\beta_2)\left[\pi^T - \underline{U}\right]^2 - \left[R-\underline{R}\right]\left[\pi^T - \underline{U}\right](\beta_1+\beta_2)\right\} \leq 0,\ f \cdot \frac{\partial L}{\partial f} = 0.$$

Then, there are four sub-cases based on whether $\frac{\partial L}{\partial f}$ and $\frac{\partial L}{\partial r}$ are equal zero, as outlined below.

<u>Case 1a</u>: Suppose $\frac{\partial L}{\partial f} = 0$ and $\frac{\partial L}{\partial r} = 0$. Since $\frac{\partial L}{\partial f} = 0$, we have $2(1-\beta_1-\beta_2)\left[\pi^T - \underline{U}\right]^2 = \left[R-\underline{R}\right]\left\{\beta_1\left[\pi^T - \underline{U}\right] + \beta_2\left[\pi^T - \underline{U}\right]\right\}$ by (A3). Substituting it into $\frac{\partial L}{\partial r} = 0$ in (A2) yields $(1-\beta_1-\beta_2)\left[\pi^T - \underline{U}\right]^2\left\{\frac{\partial R}{\partial r} - \frac{4q^T}{2+b}\right\} = 0$. This then implies $\frac{2(1-\beta_1-\beta_2)\left[b(a-c) - 2(1+b)r\right]}{(2+b)^2}\left[\pi^T - \underline{U}\right]^2 = 0$, and thus $r^T = \frac{b(a-c)}{2(1+b)} > 0$. At r^T, the operators' profit is $\pi^T = \left(q^T\right)^2 - f = \frac{(a-c)^2}{4(1+b)^2} - f$ and the port authority's fee revenue equals

$R=\dfrac{2b(a-c)^2}{4(1+b)^2}+2f$. Substituting them into $\dfrac{\partial L}{\partial f}=0$ in (A3) yields

$f^T=\dfrac{[1-(1+b)(\beta_1+\beta_2)](a-c)^2}{4(1+b)^2}-(1-\beta_1-\beta_2)\underline{U}+\dfrac{1}{2}(\beta_1+\beta_2)\underline{R}$, and (A5) and $\lambda_1^*=0$ imply

$\delta^T\in[0,\delta_1]$ with $\delta_1=\dfrac{(a-c)}{2(1+b)}$. At (r^T,f^T), we have $q^T=\dfrac{(a-c)}{2(1+b)}$ and

$\pi^T=(q^T)^2-f^T=\dfrac{(\beta_1+\beta_2)(a-c)^2}{4(1+b)}+(1-\beta_1-\beta_2)\underline{U}-\dfrac{1}{2}(\beta_1+\beta_2)\underline{R}$.

We still need to verify whether all the constraints specified in (A1) are satisfied. Upon performing some calculations, we obtain

$\bar{r}-r^T=\dfrac{(2+b)(a-c)}{2(1+b)}>0$;

$f^T>0$ if $\beta_1+\beta_2\le\dfrac{1}{1+b}$ and $\underline{U}<\underline{U}'$, or if $\beta_1+\beta_2>\dfrac{1}{1+b}$, $\underline{U}<\underline{U}'$, and

$$\underline{R}>\frac{[(1+b)(\beta_1+\beta_2)-1](a-c)^2}{2(1+b)^2(\beta_1+\beta_2)}; \tag{A9}$$

$$\pi^T-\underline{U}=\frac{(\beta_1+\beta_2)(a-c)^2}{4(1+b)}-(\beta_1+\beta_2)\underline{U}-\frac{1}{2}(\beta_1+\beta_2)\underline{R}>0 \text{ iff } \underline{U}<\underline{U}'' \text{ and } \underline{R}<\frac{(a-c)^2}{2(1+b)}; \tag{A10}$$

$$R^T-\underline{R}=\frac{(1-\beta_1-\beta_2)(a-c)^2}{2(1+b)}-2(1-\beta_1-\beta_2)\underline{U}-(1-\beta_1-\beta_2)\underline{R}>0 \text{ iff } \underline{U}<\underline{U}'' \text{ and } \underline{R}<\frac{(a-c)^2}{2(1+b)}, \tag{A11}$$

where

$$\underline{U}'\equiv\frac{[1-(1+b)(\beta_1+\beta_2)](a-c)^2}{4(1+b)^2(1-\beta_1-\beta_2)}+\frac{(\beta_1+\beta_2)\underline{R}}{2(1-\beta_1-\beta_2)} \text{ and } \underline{U}''\equiv\frac{(a-c)^2}{4(1+b)}-\frac{1}{2}\underline{R}. \tag{A12}$$

Comparing $\underline{U}'$ and $\underline{U}''$ in (A12) results in

$$\underline{U}''-\underline{U}'=\frac{b(a-c)^2}{4(1+b)^2(1-\beta_1-\beta_2)}-\frac{\underline{R}}{2(1-\beta_1-\beta_2)}\ge(<)\,0 \text{ iff } \underline{R}\le(>)\,\frac{b(a-c)^2}{2(1+b)^2}. \tag{A13}$$

Furthermore, comparing the three thresholds of $\underline{R}$ yields

$$\frac{[(1+b)(\beta_1+\beta_2)-1](a-c)^2}{2(1+b)^2(\beta_1+\beta_2)}<\frac{b(a-c)^2}{2(1+b)^2}<\frac{(a-c)^2}{2(1+b)}. \tag{A14}$$

Based on (A13)-(A14), we have $f^T > 0$, $\pi^T > \underline{U}$ and $R^T > \underline{R}$ implied by (A9)-(A11) if

$$\max\left\{0, \frac{\left[(1+b)(\beta_1+\beta_2)-1\right](a-c)^2}{2(1+b)^2(\beta_1+\beta_2)}\right\} < \underline{R} \le \frac{b(a-c)^2}{2(1+b)^2}$$ [1] and $\underline{U} < \underline{U}'$;

$$\text{or } \frac{b(a-c)^2}{2(1+b)^2} < \underline{R} < \frac{(a-c)^2}{2(1+b)} \text{ and } \underline{U} < \underline{U}''. \qquad \text{(A15)}$$

Given the conditions outlined in (A15), the optimal contract is the two-part tariff scheme denoted by

$$T^1 = \left\{ r^T = \frac{b(a-c)}{2(1+b)}, \delta^T \in \left[0, \frac{a-c}{2(1+b)}\right], f^T = \frac{\left[2-2(1+b)(\beta_1+\beta_2)\right](a-c)^2}{8(1+b)^2} - (1-\beta_1-\beta_2)\underline{U} + \frac{1}{2}(\beta_1+\beta_2)\underline{R} \right\} \text{(A16)}$$

with the net payoffs of the operators and the port authority being

$$(\pi^T - \underline{U}) = \frac{(\beta_1+\beta_2)(a-c)^2}{4(1+b)} - (\beta_1+\beta_2)\underline{U} - \frac{1}{2}(\beta_1+\beta_2)\underline{R} \text{ and}$$

$$(R^T - \underline{R}) = \frac{(1-\beta_1-\beta_2)(a-c)^2}{2(1+b)} - 2(1-\beta_1-\beta_2)\underline{U} - (1-\beta_1-\beta_2)\underline{R}, \text{ respectively.}$$

<u>Case 1b</u>: Suppose $\frac{\partial L}{\partial f} < 0$ and $\frac{\partial L}{\partial r} = 0$.

Since $\frac{\partial L}{\partial f} < 0$, we have $f^T = 0$ by (A3). At $f^T = 0$, (A2) and $\frac{\partial L}{\partial r} = 0$ imply

$(1-\beta_1-\beta_2)\frac{\partial R}{\partial r}\left[\pi^T - \underline{U}\right] - \frac{2q^T(\beta_1+\beta_2)\left[R-\underline{R}\right]}{2+b} = 0$. This, in turn, implies $r^T \in \arg\{H=0\}$,

where

$$H \equiv (1-\beta_1-\beta_2)\left[\frac{2(2+b)q^T - 2r}{2+b}\right]\left[\left(q^T\right)^2 - \underline{U}\right] - \frac{2q^T(\beta_1+\beta_2)\left[R-\underline{R}\right]}{2+b} \quad \text{with}$$

$$\frac{\partial H}{\partial r} = (1-\beta_1-\beta_2)\left\{\frac{-2(2+b)-2(2+b)}{(2+b)^2}\left[\left(q^T\right)^2 - \underline{U}\right] - \frac{2q^T\left[2(2+b)q^T - 2r\right]}{(2+b)^2}\right\}$$

$$-\frac{2(\beta_1+\beta_2)}{2+b}\left[-\frac{2rq^T - \underline{R}}{2+b} + \frac{2q^T\left[(2+b)q^T - r\right]}{2+b}\right].$$

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[1]We have $0 < \underline{R} \le \frac{b(a-c)^2}{2(1+b)^2}$ if $\beta_1+\beta_2 < \frac{1}{1+b}$; and $\frac{\left[(1+b)(\beta_1+\beta_2)-1\right](a-c)^2}{2(1+b)^2(\beta_1+\beta_2)} < \underline{R} \le \frac{b(a-c)^2}{2(1+b)^2}$ if $\beta_1+\beta_2 \ge \frac{1}{1+b}$.

We still need to verify whether $r^T > 0$ and $\frac{\partial L}{\partial f} < 0$ hold, as well as whether all the other constraints in (A1) are satisfied. First, let's examine $r^T > 0$. Upon some calculations, we find $H > 0$ and $\frac{\partial H}{\partial r} < 0$ at $r = 0$ if $\underline{U} < \underline{U}'''$;[2] and $H = 0$ and $\frac{\partial H}{\partial r} < 0$ at $r = r^T$,[3] where

$$\underline{U}''' \equiv \frac{(a-c)^2}{(2+b)^2}. \tag{A17}$$

These findings subsequently imply $r^T > 0$ if $\underline{U} < \underline{U}'''$.

Second, we examine $\frac{\partial L}{\partial f} < 0$. As demonstrated in (A3), we have

$\frac{\partial L}{\partial f} = D \cdot \left\{ 2(1-\beta_1-\beta_2)\left[\pi^T - \underline{U}\right]^2 - \left[R - \underline{R}\right]\left[\pi^T - \underline{U}\right](\beta_1+\beta_2) \right\}$

$= D \cdot \left[\pi^T - \underline{U}\right] \cdot \left\{ 2(1-\beta_1-\beta_2)\left[\pi^T - \underline{U}\right] - \left[R - \underline{R}\right](\beta_1+\beta_2) \right\}$, which implies $\frac{\partial L}{\partial f} < 0$ iff $\left[\pi^T - \underline{U}\right] < 0$[4] or $\left\{ 2(1-\beta_1-\beta_2)\left[\pi^T - \underline{U}\right] - \left[R - \underline{R}\right](\beta_1+\beta_2) \right\} < 0$, but not both. Suppose $\left\{ 2(1-\beta_1-\beta_2)\left[\pi^T - \underline{U}\right] - \left[R - \underline{R}\right](\beta_1+\beta_2) \right\} < 0$ and $\left[\pi^T - \underline{U}\right] > 0$ hold. Since $\frac{\partial L}{\partial r} = 0$, we have $(\beta_1+\beta_2)\left[\left(q^T\right)^2 - \underline{U}\right]\left[R - \underline{R}\right] = \frac{2+b}{2q^T}(1-\beta_1-\beta_2)\frac{\partial R}{\partial r}\left[\left(q^T\right)^2 - \underline{U}\right]\left[\left(q^T\right)^2 - \underline{U}\right]$. Substituting it into (A3) yields

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[2]This is because at $r = 0$, $q^T = \frac{a-c}{2+b}$, $H = 2(1-\beta_1-\beta_2)q^T\left[\left(q^T\right)^2 - \underline{U}\right] + \frac{2q^T(\beta_1+\beta_2)\underline{R}}{2+b} > 0$ if $\left(q^T\right)^2 > \underline{U}$, and

$$\frac{\partial H}{\partial r} = -\frac{4(1-\beta_1-\beta_2)}{(2+b)}\left[\left(q^T\right)^2 - \underline{U}\right] - \frac{4(1-\beta_1-\beta_2)\left(q^T\right)^2}{(2+b)} - \frac{2(\beta_1+\beta_2)\left[2(2+b)\left(q^T\right)^2 + \underline{R}\right]}{(2+b)^2} < 0 \text{ if } \left(q^T\right)^2 > \underline{U}.$$

Furthermore, $\left(q^T\right)^2 > \underline{U}$ holds if and only if $\underline{U} < \underline{U}'''$.

[3]This is because $H = 0$ at $r = r^T$ by definition, thereby implying

$$\frac{\partial H}{\partial r} = -\frac{4(2+b)(1-\beta_1-\beta_2)}{(2+b)^2}\left[\left(q^T\right)^2 - \underline{U}\right] - \frac{4(\beta_1+\beta_2)q^T\left[(2+b)q^T - r^T\right]}{(2+b)^2} - \frac{(1-\beta_1-\beta_2)\left[2(2+b)q^T - 2r^T\right]}{(2+b)^2 q^T}\left[\left(q^T\right)^2 + \underline{U}\right] < 0.$$

The proofs are available upon request.

[4]Considering $\left[\pi^T - \underline{U}\right] < 0$ here involves discussing the situation not at equilibria. However, we will still verify $\left[\pi^T - \underline{U}\right] > 0$ at equilibria.

$\frac{\partial L}{\partial f}=\frac{(1-\beta_1-\beta_2)}{(2+b)q^T}\left[\left(q^T\right)^2-\underline{U}\right]^2\left[2(1+b)r-b(a-c)\right]<0$ if $r<\frac{b(a-c)}{2(1+b)}$. However, we still need to show $r^T<\frac{b(a-c)}{2(1+b)}$ as follows. By $\left(q^T\right)^2>\underline{U}$ and $q^T=\frac{(a-c-r)}{2+b}$, subsequent calculations result in

$$\frac{\partial H}{\partial r}=\frac{-4\left[\left(2-\beta_1-\beta_2\right)(a-c)-3r\right]q^T}{(2+b)^2}+\frac{4(1-\beta_1-\beta_2)\underline{U}}{(2+b)}-\frac{2\left(\beta_1+\beta_2\right)\underline{R}}{(2+b)^2}$$

$$<\frac{-4\left[(a-c)-\left(2+\beta_1+\beta_2\right)r\right]q^T}{(2+b)^2}-\frac{2\left(\beta_1+\beta_2\right)\underline{R}}{(2+b)^2}.$$

Thus, we have $\frac{\partial H}{\partial r}<0$ if $r<\frac{b(a-c)}{2(1+b)}$. On the other hand, we have $H>0$ at $r=0$ as shown in footnote 2 by $\left(q^T\right)^2>\underline{U}$, and $H<0$ at $r=\frac{b(a-c)}{2(1+b)}$ iff $\underline{U}>\underline{U}'$ defined in (A12).[5] Since $H=0$ at $r=r^T$, we must have $r^T<\frac{b(a-c)}{2(1+b)}$ if $\underline{U}>\underline{U}'$ and $\left(q^T\right)^2>\underline{U}$. These, in turn, imply $\frac{\partial L}{\partial f}<0$ if $\underline{U}>\underline{U}'$ and $\left(q^T\right)^2>\underline{U}$.

Alternatively, supposing $\left[\pi^T-\underline{U}\right]<0$ and

$\left\{2(1-\beta_1-\beta_2)\left[\pi^T-\underline{U}\right]-\left[R-\underline{R}\right]\left(\beta_1+\beta_2\right)\right\}>0$ hold. Under these conditions, we have

$$f<\frac{\left(a-c-r^T\right)\left[\left(1-\beta_1-\beta_2\right)(a-c)-\left[1+(1+b)\left(\beta_1+\beta_2\right)\right]r^T\right]}{(2+b)^2}-\left(1-\beta_1-\beta_2\right)\underline{U}+\frac{1}{2}\left(\beta_1+\beta_2\right)\underline{R}$$

$$<\frac{\left[1-(1+b)\left(\beta_1+\beta_2\right)\right](a-c)^2}{4(1+b)^2}-\left(1-\beta_1-\beta_2\right)\underline{U}+\frac{1}{2}\left(\beta_1+\beta_2\right)\underline{R}$$ by

$$\left\{2(1-\beta_1-\beta_2)\left[\pi^T-\underline{U}\right]-\left[R-\underline{R}\right]\left(\beta_1+\beta_2\right)\right\}$$

$$=\left\{\frac{2\left(a-c-r^T\right)\left[\left(1-\beta_1-\beta_2\right)(a-c)-\left[1+(1+b)\left(\beta_1+\beta_2\right)\right]r^T\right]}{(2+b)^2}-2\left(1-\beta_1-\beta_2\right)\underline{U}+\left(\beta_1+\beta_2\right)\underline{R}\right\}.$$ These

suggest that the conditions for $\left[\pi^T-\underline{U}\right]<0$ are equivalent to those for $\left[R-\underline{R}\right]<0$ because

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[5] When $r=\frac{b(a-c)}{2(1+b)}$, we have $q^T=\frac{a-c}{2(1+b)}$, $R=\frac{b(a-c)^2}{2(1+b)^2}$ and $H=\left(1-\beta_1-\beta_2\right)\left[\frac{2(2+b)q^T-2r}{2+b}\right]\left[\left(q^T\right)^2-\underline{U}\right]-\frac{2q^T\left(\beta_1+\beta_2\right)\left[R-\underline{R}\right]}{2+b}$

$=\frac{(a-c)}{(1+b)(2+b)}\left\{\frac{\left[1-(1+b)\left(\beta_1+\beta_2\right)\right](a-c)^2}{2(1+b)^2}-2\left(1-\beta_1-\beta_2\right)\underline{U}+\left(\beta_1+\beta_2\right)\underline{R}\right\}<(>)\,0$ iff

$\underline{U}>(<)\frac{\left[1-(1+b)\left(\beta_1+\beta_2\right)\right](a-c)^2}{4(1+b)^2\left(1-\beta_1-\beta_2\right)}+\frac{\left(\beta_1+\beta_2\right)\underline{R}}{2\left(1-\beta_1-\beta_2\right)}\equiv\underline{U}'$. This implies $H<0$ at $r=\frac{b(a-c)}{2(1+b)}$ iff $\underline{U}>\underline{U}'$.

$\left\{2(1-\beta_1-\beta_2)\left[\pi^T-\underline{U}\right]-\left[R-\underline{R}\right](\beta_1+\beta_2)\right\}>0$ and $(\beta_1+\beta_2)\left[R-\underline{R}\right]=\frac{2+b}{2q^T}(1-\beta_1-\beta_2)\frac{\partial R}{\partial r}\left[\left(q^T\right)^2-\underline{U}\right]$ by $\frac{\partial L}{\partial r}=0$. Furthermore, we have

$$\left[R-\underline{R}\right]=\frac{2r^T\left(a-c-r^T\right)}{(2+b)}+2f-\underline{R}$$

$$<\frac{2r^T\left(a-c-r^T\right)}{(2+b)}+\frac{2\left(a-c-r^T\right)\left[(1-\beta_1-\beta_2)(a-c)-\left[1+(1+b)(\beta_1+\beta_2)\right]r^T\right]}{(2+b)^2}-2(1-\beta_1-\beta_2)\underline{U}+(\beta_1+\beta_2)\underline{R}-\underline{R}$$

$$=\frac{2(1-\beta_1-\beta_2)\left(a-c-r^T\right)\left[(a-c)+(1+b)r^T\right]}{(2+b)^2}-2(1-\beta_1-\beta_2)\underline{U}-(1-\beta_1-\beta_2)\underline{R}$$

$<\frac{(1-\beta_1-\beta_2)(a-c)^2}{2(1+b)}-2(1-\beta_1-\beta_2)\underline{U}-(1-\beta_1-\beta_2)\underline{R}<0$ iff $\underline{U}>\underline{U}''$ and $\underline{R}<\frac{(a-c)^2}{2(1+b)}$, or iff $\underline{R}>\frac{(a-c)^2}{2(1+b)}$. These consequently imply $\frac{\partial L}{\partial f}<0$ if $\underline{U}>\underline{U}''$ and $\underline{R}<\frac{(a-c)^2}{2(1+b)}$ or if $\underline{R}>\frac{(a-c)^2}{2(1+b)}$.

Third, to ensure consistency between conditions $\underline{U}>\underline{U}'$ and $\underline{U}<\underline{U}'''$, we must have $\underline{U}'<\underline{U}'''$. Subsequent calculations yield

$$\underline{U}'-\underline{U}'''=\frac{(a-c)^2\left[-b(4+3b)-b^2(1+b)(\beta_1+\beta_2)\right]}{4(1+b)^2(2+b)^2(1-\beta_1-\beta_2)}+\frac{(\beta_1+\beta_2)\underline{R}}{2(1-\beta_1-\beta_2)}<(>)\,0$$

$$\text{iff }\ \underline{R}<(>)\,\frac{b(a-c)^2\left[(4+3b)+b(1+b)(\beta_1+\beta_2)\right]}{2(1+b)^2(2+b)^2(\beta_1+\beta_2)}.$$

Furthermore, we can obtain

$$\frac{b(a-c)^2}{2(1+b)^2}<\frac{(a-c)^2}{2(1+b)}<\frac{b(a-c)^2\left[(4+3b)+b(1+b)(\beta_1+\beta_2)\right]}{2(1+b)^2(2+b)^2(\beta_1+\beta_2)}.$$

Fourth, we verify whether all the constraints specified in (A1) are met. Given $r^T<\frac{b(a-c)}{2(1+b)}$, we have $\bar{r}-r^T>\bar{r}-\frac{b(a-c)}{2(1+b)}=\frac{(2+b)(a-c)}{2(1+b)}>0$. On the other hand, as shown in footnote 2, we have $\pi^T>\underline{U}$ if and only if $\underline{U}<\underline{U}'''$. Furthermore, since $H=(1-\beta_1-\beta_2)\left[\frac{2(2+b)q^T-2r^T}{2+b}\right]\left[\left(q^T\right)^2-\underline{U}\right]-\frac{2q^T(\beta_1+\beta_2)\left[R-\underline{R}\right]}{2+b}=0$ at $r=r^T$, $\left[\left(q^T\right)^2-\underline{U}\right]>0$, and $\left[\dfrac{2(2+b)q^T-2r^T}{2+b}\right]=\dfrac{2(a-2r^T-c)}{2+b}>0$ by $\dfrac{2(a-2r^T-c)}{2+b}>\dfrac{2(a-c)}{(1+b)(2+b)}>0$ and $r^T<\frac{b(a-c)}{2(1+b)}$, we have $\dfrac{2q^T(\beta_1+\beta_2)[R-\underline{R}]}{2+b}>0$. This, in turn, implies $R>\underline{R}$ if r^T exists.

Consequently, under the conditions of

$$\min\{\underline{U}'',\underline{U}'\}<\underline{U}<\underline{U}''' \text{ and } \underline{R}<\frac{b(a-c)^2[(4+3b)+b(1+b)(\beta_1+\beta_2)]}{2(1+b)^2(2+b)^2(\beta_1+\beta_2)}, \quad \text{(A18)}$$

the optimal contract is the unit-fee scheme denoted by

$$U^1=\left\{r^T\in\arg\left\{\frac{2(1-\beta_1-\beta_2)(a-2r-c)}{2+b}\left[\frac{(a-r-c)^2}{(2+b)^2}-\underline{U}\right]-\frac{2(\beta_1+\beta_2)(a-r-c)}{(2+b)^2}\left[\frac{2r(a-r-c)}{2+b}-\underline{R}\right]=0\right\},\delta^T\in\left[0,\frac{a-c-r^T}{2-b}\right],f^T=0\right\}$$

(A19)

As we lack the analytical solution for r^T, we're unable to determine the net payoffs of operators and the port authority as seen in Case 1a.

Case 1c: Suppose $\dfrac{\partial L}{\partial f}=0$ and $\dfrac{\partial L}{\partial r}<0$. Since $\dfrac{\partial L}{\partial r}<0$, we have $r^T=0$. Substituting it into $\dfrac{\partial L}{\partial f}=0$ results in $f^T=\dfrac{[1-(\beta_1+\beta_2)](a-c)^2}{(2+b)^2}-(1-\beta_1-\beta_2)\underline{U}+\dfrac{1}{2}(\beta_1+\beta_2)\underline{R}$. Substituting f^T into $\dfrac{\partial L}{\partial r}$ yields $\dfrac{\partial L}{\partial r}=(1-\beta_1-\beta_2)[\pi^T-\underline{U}]^2\left\{\dfrac{\partial R}{\partial r}-\dfrac{4q^T}{2+b}\right\}=\dfrac{2b(a-c)}{(2+b)^2}(1-\beta_1-\beta_2)[\pi^T-\underline{U}]^2>0$. This contradicts $\dfrac{\partial L}{\partial r}<0$, thus there is no solution in this scenario.

Case 1d: Suppose $\dfrac{\partial L}{\partial f}<0$ and $\dfrac{\partial L}{\partial r}<0$. Both $\dfrac{\partial L}{\partial f}<0$ and $\dfrac{\partial L}{\partial r}<0$ imply $r^T=0$ and $f^T=0$, consequently, no meaningful solution exists.

Case 2: Suppose $\delta\in\left(\delta_1,\dfrac{a}{1+b}\right)$ with $\delta_1\equiv\dfrac{a-r-c}{2+b}$. We have $q_1^T=q_2^T=\delta$, $\pi_1^T=\pi_2^T=\{\delta[a-(1+b)\delta-c-r]-f\}\equiv\pi^T$ and $R=2r\delta+2f$ by Lemma 1. Hence, the issue in equation (7) transforms into

$$\max_{r\geq 0,f\geq 0,\delta\geq 0}\ W\equiv[R-\underline{R}]^{(1-\beta_1-\beta_2)}[\pi^T-\underline{U}]^{\beta_1}[\pi^T-\underline{U}]^{\beta_2}$$

$$\text{s.t. } \delta_1<\delta<\frac{a}{1+b},\ 0\leq r<\overline{r},\ f\geq 0,\ \pi^T\geq\underline{U} \text{ and } 2r\delta+2f\geq\underline{R}. \quad \text{(A20)}$$

The Lagrange function associated with this problem is

$$L=\begin{Bmatrix}\left[2r\delta+2f-\underline{R}\right]^{(1-\beta_1-\beta_2)}\left[\pi^T-\underline{U}\right]^{\beta_1}\left[\pi^T-\underline{U}\right]^{\beta_2}+\lambda_1\left(\delta-\delta_1\right)+\lambda_2\left(\frac{a}{1+b}-\delta\right)\\ +\lambda_3\left(\overline{r}-r\right)+\lambda_4\left(\pi^T-\underline{U}\right)+\lambda_5\left[2r\delta+2f-\underline{R}\right]\end{Bmatrix},$$

where λ_1, λ_2, λ_3, λ_4 and λ_5 are respective Lagrange multipliers associated with the five inequality constraints in (A20) except for $r\geq 0$ and $f\geq 0$. The corresponding Kuhn-Tucker conditions are as follows.

$$\frac{\partial L}{\partial r}=\begin{Bmatrix}D\cdot\left\{2(1-\beta_1-\beta_2)\delta\left[\pi-\underline{U}\right]^2-\left(\beta_1+\beta_2\right)\delta\left[R-\underline{R}\right]\left[\pi-\underline{U}\right]\right\}\\ +\frac{1}{2+b}\lambda_1-\lambda_2-\lambda_3-\frac{2q^T}{2+b}\lambda_4+\frac{2(a-2r)-(c_1+c_2)}{2+b}\lambda_5\end{Bmatrix}\leq 0,\ r\cdot\frac{\partial L}{\partial r}=0,$$

$$\frac{\partial L}{\partial f}=\left\{D\cdot\left\{2(1-\beta_1-\beta_2)\left[\pi-\underline{U}\right]^2-\left(\beta_1+\beta_2\right)\left[R-\underline{R}\right]\left[\pi-\underline{U}\right]\right\}-\lambda_4+2\lambda_5\right\}\leq 0,\ f\cdot\frac{\partial L}{\partial f}=0,$$

$$\frac{\partial L}{\partial \delta}=\begin{Bmatrix}D\cdot\left\{(1-\beta_1-\beta_2)\frac{\partial R}{\partial\delta}\left[\pi-\underline{U}\right]^2+\left(\beta_1+\beta_2\right)\frac{\partial\pi^T}{\partial\delta}\left[R-\underline{R}\right]\left[\pi-\underline{U}\right]\right\}\\ +\lambda_1-\lambda_2+\left[a-2\left(1+b\right)\delta-r-c\right]\lambda_4+2r\lambda_5\end{Bmatrix}\leq 0,\ \delta\cdot\frac{\partial L}{\partial\delta}=0,$$

$$\frac{\partial L}{\partial\lambda_1}=\delta-\left[\frac{a-r}{2+b}+\frac{bc_1-2c_2}{4-b^2}\right]\geq 0,\ \lambda_1\cdot\frac{\partial L}{\partial\lambda_1}=0,$$

$$\frac{\partial L}{\partial\lambda_2}=\frac{a}{1+b}-\delta\geq 0,\ \lambda_2\cdot\frac{\partial L}{\partial\lambda_2}=0,$$

$$\frac{\partial L}{\partial\lambda_3}=\overline{r}-r\geq 0,\ \lambda_3\cdot\frac{\partial L}{\partial\lambda_3}=0.$$

$$\frac{\partial L}{\partial\lambda_4}=\pi^T-\underline{U}\geq 0,\ \lambda_4\cdot\frac{\partial L}{\partial\lambda_4}=0,\ \text{and}$$

$$\frac{\partial L}{\partial\lambda_5}=2\delta r+2f-\underline{R}\geq 0,\ \lambda_5\cdot\frac{\partial L}{\partial\lambda_5}=0,$$

where $D=\left\{[R-\underline{R}]^{-(\beta_1+\beta_2)}[\pi^T-\underline{U}]^{-(2-\beta_1-\beta_2)}\right\}>0$.

As previously mentioned, we establish $\lambda_4^*=\lambda_5^*=0$ by considering the cases of $\pi^T>\underline{U}$ and $R>\underline{R}$. Additionally, the constraints of $\delta_1<\delta<\frac{a}{1+b}$ and $0\leq r<\overline{r}$ imply $\lambda_1^*=\lambda_2^*=\lambda_3^*=0$. Consequently, the signs of $\frac{\partial L}{\partial r}$ and $\frac{\partial L}{\partial f}$ are the same, given that

$$\frac{\partial L}{\partial r}=D\cdot\delta\left\{2(1-\beta_1-\beta_2)\left[\pi-\underline{U}\right]^2-\left(\beta_1+\beta_2\right)\left[R-\underline{R}\right]\left[\pi-\underline{U}\right]\right\},$$

$$\frac{\partial L}{\partial f}=D\cdot\left\{2(1-\beta_1-\beta_2)\left[\pi-\underline{U}\right]^2-\left(\beta_1+\beta_2\right)\left[R-\underline{R}\right]\left[\pi-\underline{U}\right]\right\},$$

and $\frac{\partial L}{\partial r}=\delta\cdot\frac{\partial L}{\partial f}$. These imply $\frac{\partial L}{\partial r}<0$ iff $\frac{\partial L}{\partial f}<0$ by $\delta>0$, thus $r^T=f^T=0$. Given that, this case is not intriguing, so we shift our attention to the scenario of $\frac{\partial L}{\partial r}=\frac{\partial L}{\partial f}=0$. Therefore, we have $\left\{2(1-\beta_1-\beta_2)\left[\pi^T-\underline{U}\right]-\left(\beta_1+\beta_2\right)\left[R-\underline{R}\right]\right\}=0$. Substituting it into $\frac{\partial L}{\partial \delta}=0$ results in $\frac{\partial L}{\partial \delta}=(1-\beta_1-\beta_2)\left[\pi^T-\underline{U}\right]^2\left\{\frac{\partial R}{\partial \delta}+2\frac{\partial \pi}{\partial \delta}\right\}=2(1-\beta_1-\beta_2)\left[\pi^T-\underline{U}\right]^2\left\{a-2\left(1+b\right)\delta-c\right\}=0$, hence $\delta^T=\frac{a-c}{2\left(1+b\right)}>0$. Then, substituting δ^T into $\frac{\partial L}{\partial f}=0$ yields

$$f^T=\frac{\left[1-\left(\beta_1+\beta_2\right)\right]\left(a-c\right)^2}{4\left(1+b\right)}-\frac{\left(a-c\right)r^T}{2\left(1+b\right)}-\left(1-\beta_1-\beta_2\right)\underline{U}+\frac{1}{2}\left(\beta_1+\beta_2\right)\underline{R}$$

with

$$f^T\geq 0 \text{ iff } r^T\leq\frac{\left(1-\beta_1-\beta_2\right)\left(a-c\right)}{2}-\frac{2\left(1+b\right)\left(1-\beta_1-\beta_2\right)}{\left(a-c\right)}\underline{U}+\frac{\left(1+b\right)\left(\beta_1+\beta_2\right)}{\left(a-c\right)}\underline{R}. \quad \text{(A21)}$$

We need to verify whether all the other constraints in (A20) are satisfied. Initially, we must ensure that the constraint $\delta_1<\delta^T<\frac{a}{1+b}$ is met, necessitating the condition $r^T>\frac{b\left(a-c\right)}{2\left(1+b\right)}$.[6]

Conversely, $f^T>0$ imposes an upper limit on r^T. Therefore, we need to access the compatibility between these two conditions presented below. Through some calculations, we find that

$$\frac{\left(1-\beta_1-\beta_2\right)\left(a-c\right)}{2}-\frac{2\left(1+b\right)\left(1-\beta_1-\beta_2\right)}{\left(a-c\right)}\underline{U}+\frac{\left(1+b\right)\left(\beta_1+\beta_2\right)}{\left(a-c\right)}\underline{R}-\frac{b\left(a-c\right)}{2\left(1+b\right)}$$

$$=\frac{\left[1-\left(1+b\right)\left(\beta_1+\beta_2\right)\right]\left(a-c\right)^2+2\left(1+b\right)^2\left(\beta_1+\beta_2\right)\underline{R}-4\left(1+b\right)^2\left(1-\beta_1-\beta_2\right)\underline{U}}{2\left(1+b\right)\left(a-c\right)}>0 \text{ iff } \underline{U}<\underline{U}',$$

where $\underline{U}'$ is defined in (A12) with

[6]The condition of $\delta_1<\delta^T$ is equivalent to $\frac{a-c-r^T}{2+b}<\frac{a-c}{2(1+b)}$, a consequence of $r^T>\frac{(a-c)b}{2(1+b)}$.

$$\underline{U}' > 0 \text{ if } \beta_1+\beta_2 \le \frac{1}{1+b}, \text{ or if } \beta_1+\beta_2 > \frac{1}{1+b} \text{ and } \underline{R} > \frac{\left[(1+b)(\beta_1+\beta_2)-1\right](a-c)^2}{2(1+b)^2(\beta_1+\beta_2)}. \quad \text{(A22)}$$

Next, we verify $\pi^T > \underline{U}$ and $R^T \ge \underline{R}$. After some calculations, it is evident that

$q^T = \delta^T = \frac{(a-c)}{2(1+b)}$,

$\pi^T = \left[a-(1+b)\delta^T - r^T - c\right]\delta^T - f^T = \frac{(\beta_1+\beta_2)(a-c)^2}{4(1+b)} + (1-\beta_1-\beta_2)\underline{U} - \frac{1}{2}(\beta_1+\beta_2)\underline{R}$,

$\pi^T - \underline{U} = \frac{(\beta_1+\beta_2)(a-c)^2}{4(1+b)} - (\beta_1+\beta_2)\underline{U} - \frac{1}{2}(\beta_1+\beta_2)\underline{R}$, and

$R^T - \underline{R} = \frac{(1-\beta_1-\beta_2)(a-c)^2}{2(1+b)} - 2(1-\beta_1-\beta_2)\underline{U} - (1-\beta_1-\beta_2)\underline{R}$.

Therefore, as in (A10) and (A11), we have $\pi^T > \underline{U}$ and $R^T > \underline{R}$ iff $\underline{U} < \underline{U}''$ and $\underline{R} < \frac{(a-c)^2}{2(1+b)}$

Third, we verify $r < \bar{r}$. Calculations indicate

$$\bar{r} - r^T > \bar{r} - \left\{\frac{(1-\beta_1-\beta_2)(a-c)}{2} - \frac{2(1+b)(1-\beta_1-\beta_2)}{(a-c)}\underline{U} + \frac{(1+b)(\beta_1+\beta_2)}{(a-c)}\underline{R}\right\}$$

$$= \left\{\frac{(1+\beta_1+\beta_2)(a-c)}{2} + \frac{2(1+b)(1-\beta_1-\beta_2)}{(a-c)}\underline{U} - \frac{(1+b)(\beta_1+\beta_2)}{(a-c)}\underline{R}\right\}$$

$$> \left\{\frac{(1+\beta_1+\beta_2)(a-c)}{2} + \frac{2(1+b)(1-\beta_1-\beta_2)}{(a-c)}\underline{U} - \frac{(1+b)(\beta_1+\beta_2)}{(a-c)} \times \frac{(a-c)^2}{2(1+b)}\right\}$$

$$= \left\{\frac{(a-c)}{2} + \frac{2(1+b)(1-\beta_1-\beta_2)}{(a-c)}\underline{U}\right\} > 0 \quad \text{(A23)}$$

by $\underline{R} < \frac{(a-c)^2}{2(1+b)}$.

As in Case 1a, it follows that (A14) implies that (A10), (A11), (A22) and (A23) will be held when (A15) holds. Therefore, under (A15), the constraints of $\delta_1 < \delta^T < \frac{a}{1+b}$, $0 \le r < \bar{r}$, $\pi^T > \underline{U}$ and $R^T > \underline{R}$ will be satisfied. Consequently, under the conditions specified in (A15), (A21) suggests two potential optimal contracts. The first optimal contract is the two-part tariff scheme represented by

$$T^2 = \left\{ \begin{array}{l} r^T \in \left(\frac{b(a-c)}{2(1+b)}, \frac{(1-\beta_1-\beta_2)(a-c)^2 - 4(1+b)(1-\beta_1-\beta_2)\underline{U} + 2(1+b)(\beta_1+\beta_2)\underline{R}}{2(a-c)} \right), \ \delta^T = \frac{a-c}{2(1+b)}, \\ f^T = \frac{[1-(\beta_1+\beta_2)](a-c)^2 - 2(a-c)r^T}{4(1+b)} - (1-\beta_1-\beta_2)\underline{U} + \frac{1}{2}(\beta_1+\beta_2)\underline{R} \end{array} \right\}.$$

(A24)

The second potential optimal contract is the unit-fee scheme indicated by

$$U^2 = \left\{ r^T = \frac{(1-\beta_1-\beta_2)(a-c)}{2} - \frac{2(1+b)(1-\beta_1-\beta_2)}{(a-c)}\underline{U} + \frac{(1+b)(\beta_1+\beta_2)}{(a-c)}\underline{R}, \delta^T = \frac{a-c}{2(1+b)}, f^T = 0 \right\}. \quad \text{(A25)}$$

Regardless of the two-part tariff or unit-fee contract offered, the equilibrium output and profit of the operators are $\frac{a-c}{2(1+b)}$ and $\frac{(\beta_1+\beta_2)(a-c)^2}{4(1+b)} + (1-\beta_1-\beta_2)\underline{U} - \frac{1}{2}(\beta_1+\beta_2)\underline{R}$, and the port authority's equilibrium fee revenue equals

$$R = 2r\delta^T + 2f^T = \frac{(1-\beta_1-\beta_2)(a-c)^2}{2(1+b)} - 2(1-\beta_1-\beta_2)\underline{U} + (\beta_1+\beta_2)\underline{R}.$$

To summarize, the optimal contract is the two-part tariff scheme T^1 in (A16) under the conditions (A15), as demonstrated by Case 1a. Additionally, it's the unit-fee scheme U^1 in (A19) under the conditions (A18), as shown by Case 1b. Furthermore, the two-part tariff scheme T^2 in (A24) under conditions (A15) is depicted by Case 2, and the unit-fee scheme U^2 in (A25) under conditions (A15) is shown by Case 2 as well.

Finally, we'll represent the conditions provided in (A15) and (A18) in terms of the model's parameters, $a, b, c, \beta_1, \beta_2, \underline{U}$ and $\underline{R}$, as follows. Initially, comparing $\underline{U}', \underline{U}''$ and $\underline{U}'''$, defined in (A12) and (A17) respectively, results in

$$\underline{U}' < \underline{U}''' < \underline{U}'' \text{ if } \underline{R} < \frac{b^2(a-c)^2}{2(1+b)(2+b)^2},$$

$$\underline{U}' < \underline{U}'' < \underline{U}''' \text{ if } \frac{b^2(a-c)^2}{2(1+b)(2+b)^2} < \underline{R} \le \frac{b(a-c)^2}{2(1+b)^2}, \text{ and}$$

$$\underline{U}'' < \underline{U}' < \underline{U}''' \text{ if } \frac{b(a-c)^2}{2(1+b)^2} < \underline{R} < \frac{b(a-c)^2[(4+3b)+b(1+b)(\beta_1+\beta_2)]}{2(1+b)^2(2+b)^2(\beta_1+\beta_2)}.$$

Then, comparing the five critical values of $\underline{R}$, $\frac{[(1+b)(\beta_1+\beta_2)-1](a-c)^2}{2(1+b)^2(\beta_1+\beta_2)}$, $\frac{b^2(a-c)^2}{2(1+b)(2+b)^2}$, $\frac{b(a-c)^2}{2(1+b)^2}$, $\frac{(a-c)^2}{2(1+b)}$ and $\frac{b(a-c)^2[(4+3b)+b(1+b)(\beta_1+\beta_2)]}{2(1+b)^2(2+b)^2(\beta_1+\beta_2)}$ yields

$$\frac{b^2(a-c)^2}{2(1+b)(2+b)^2} < \frac{b(a-c)^2}{2(1+b)^2} < \frac{(a-c)^2}{2(1+b)} < \frac{b(a-c)^2[(4+3b)+b(1+b)(\beta_1+\beta_2)]}{2(1+b)^2(2+b)^2(\beta_1+\beta_2)}$$

for $0<\beta_1+\beta_2<\frac{b(4+3b)}{4(1+b)}$;

$$\frac{b^2(a-c)^2}{2(1+b)(2+b)^2} < \frac{b(a-c)^2}{2(1+b)^2} < \frac{b(a-c)^2[(4+3b)+b(1+b)(\beta_1+\beta_2)]}{2(1+b)^2(2+b)^2(\beta_1+\beta_2)} < \frac{(a-c)^2}{2(1+b)}$$

for $\frac{b(4+3b)}{4(1+b)}<\beta_1+\beta_2<\frac{1}{1+b}$;

$$\frac{[(1+b)(\beta_1+\beta_2)-1](a-c)^2}{2(1+b)^2(\beta_1+\beta_2)} < \frac{b^2(a-c)^2}{2(1+b)(2+b)^2} < \frac{b(a-c)^2}{2(1+b)^2} < \frac{b(a-c)^2[(4+3b)+b(1+b)(\beta_1+\beta_2)]}{2(1+b)^2(2+b)^2(\beta_1+\beta_2)} < \frac{(a-c)^2}{2(1+b)}$$

for $\frac{1}{1+b}\le\beta_1+\beta_2<\frac{(2+b)^2}{4(1+b)^2}$; and

$$\frac{b^2(a-c)^2}{2(1+b)(2+b)^2} < \frac{[(1+b)(\beta_1+\beta_2)-1](a-c)^2}{2(1+b)^2(\beta_1+\beta_2)} < \frac{b(a-c)^2}{2(1+b)^2} < \frac{b(a-c)^2[(4+3b)+b(1+b)(\beta_1+\beta_2)]}{2(1+b)^2(2+b)^2(\beta_1+\beta_2)} < \frac{(a-c)^2}{2(1+b)}$$

for $\frac{(2+b)^2}{4(1+b)^2}\le\beta_1+\beta_2<1$.

Subsequently, considering the values of a, b, c, β_1, β_2, $\underline{U}$ and $\underline{R}$, we can derive the corresponding optimal concession contracts outlined in Table 1 of Proposition 1. To illustrate, let's examine the scenario where $\beta_1+\beta_2<\frac{b(4+3b)}{4(1+b)}$ and $\underline{R}<\frac{b^2(a-c)^2}{2(1+b)(2+b)^2}$, which consequently imply $\underline{U}'<\underline{U}'''<\underline{U}''$ and

$$\underline{R} < \frac{b^2(a-c)^2}{2(1+b)(2+b)^2} < \frac{b(a-c)^2}{2(1+b)^2} < \frac{(a-c)^2}{2(1+b)} < \frac{b(a-c)^2[(4+3b)+b(1+b)(\beta_1+\beta_2)]}{2(1+b)^2(2+b)^2(\beta_1+\beta_2)}.$$

The optimality between the two-part tariff and unit-fee schemes hinge on the values of $\underline{U}$. If $\underline{U} < \underline{U}'$, then T^1, T^2 and U^2 can serve as optimal contracts according to (A15), whereas U^1 is the optimal contract for $\underline{U}' < \underline{U} < \underline{U}'''$ based on (A18). However, there exists no optimal contract for $\underline{U}''' < \underline{U} < \underline{U}''$. Similar arguments can yield other results.

Proof of Proposition 2: Given that $c_1 < c_2$, we have $\delta_1 < \delta_2$ according to Lemma 1, leading to three cases as follows.

Case 1: Suppose $\delta \in [0, \delta_1]$ with $\delta_1 \equiv \dfrac{a-r}{2+b} + \dfrac{bc_1 - 2c_2}{4-b^2} > 0$. Consequently, we have $\pi_i^T = \left[\left(q_i^T\right)^2 - f\right]$ for $i = 1, 2$ with $q_1^T \equiv \dfrac{a-r}{2+b} + \dfrac{bc_2 - 2c_1}{4-b^2}$, $q_2^T = \delta_1$ and $R = r(q_1^T + q_2^T) + 2f$. Thus, the problem in (7) transforms into

$$\max_{r\ge 0, f\ge 0, \delta \ge 0} W \equiv [R - \underline{R}]^{(1-\beta_1-\beta_2)}[\pi_1^T - \underline{U}_1]^{\beta_1}[\pi_2^T - \underline{U}_2]^{\beta_2}$$

$$\text{s.t. } 0 \le \delta \le \delta_1,\ 0 \le r < \overline{r},\ f \ge 0,\ \pi_1^T \ge \underline{U}_1,\ \pi_2^T \ge \underline{U}_2 \text{ and } r(q_1^T + q_2^T) + 2f \ge \underline{R}. \qquad \text{(A26)}$$

The associated Lagrange function is

$$L = \begin{Bmatrix} \left[R - \underline{R}\right]^{(1-\beta_1-\beta_2)} \left[\pi_1^T - \underline{U}_1\right]^{\beta_1} \left[\pi_2^T - \underline{U}_2\right]^{\beta_2} + \lambda_1\left(\delta_1 - \delta\right) + \lambda_2\left(\overline{r} - r\right) \\ + \lambda_3\left(\pi_1^T - \underline{U}_1\right) + \lambda_4\left(\pi_2^T - \underline{U}_2\right) + \lambda_5\left[r(q_1^T + q_2^T) + 2f - \underline{R}\right] \end{Bmatrix},$$

where λ_1, λ_2, λ_3, λ_4 and λ_5 are Lagrange multipliers associated with the five inequality constraints in (A26) except for $r \ge 0$, $f \ge 0$ and $\delta \ge 0$. The corresponding Kuhn-Tucker conditions are as follows.

$$\frac{\partial L}{\partial r} = \begin{Bmatrix} D \cdot \left\{ (1-\beta_1-\beta_2)\dfrac{\partial R}{\partial r}\left[\pi_1^T - \underline{U}_1\right]\left[\pi_2^T - \underline{U}_2\right] - \dfrac{2\beta_1 q_1^T}{2+b}\left[R - \underline{R}\right]\left[\pi_2^T - \underline{U}_2\right] - \dfrac{2\beta_2 q_2^T}{2+b}\left[R - \underline{R}\right]\left[\pi_1^T - \underline{U}_1\right] \right\} \\ -\dfrac{1}{2+b}\lambda_1 - \lambda_2 - \dfrac{2q_1^T}{2+b}\lambda_3 - \dfrac{2q_2^T}{2+b}\lambda_4 + \dfrac{2(a-2r) - (c_1 + c_2)}{2+b}\lambda_5 \end{Bmatrix} \le 0,\ r \cdot \frac{\partial L}{\partial r} = 0,$$

(A27)

$$\frac{\partial L}{\partial f} = \left\{ D \cdot \begin{Bmatrix} 2(1-\beta_1-\beta_2)\left[\pi_1^T - \underline{U}_1\right]\left[\pi_2^T - \underline{U}_2\right] \\ -\beta_1\left[R - \underline{R}\right]\left[\pi_2^T - \underline{U}_2\right] - \beta_2\left[R - \underline{R}\right]\left[\pi_1^T - \underline{U}_1\right] \end{Bmatrix} - \lambda_3 - \lambda_4 + 2\lambda_5 \right\} \le 0,\ f \cdot \frac{\partial L}{\partial f} = 0, \quad \text{(A28)}$$

$$\frac{\partial L}{\partial \delta} = -\lambda_1 \le 0,\ \delta \cdot \frac{\partial L}{\partial \delta} = 0, \qquad \text{(A29)}$$

$$\frac{\partial L}{\partial \lambda_1} = \frac{a-r}{2+b} + \frac{bc_1 - 2c_2}{4-b^2} - \delta \ge 0,\ \lambda_1 \cdot \frac{\partial L}{\partial \lambda_1} = 0, \qquad \text{(A30)}$$

$$\frac{\partial L}{\partial \lambda_2} = \overline{r} - r \geq 0,\ \lambda_2 \cdot \frac{\partial L}{\partial \lambda_2} = 0, \tag{A31}$$

$$\frac{\partial L}{\partial \lambda_3} = \pi_1^T - \underline{U}_1 \geq 0,\ \lambda_3 \cdot \frac{\partial L}{\partial \lambda_3} = 0, \tag{A32}$$

$$\frac{\partial L}{\partial \lambda_4} = \pi_2^T - \underline{U}_2 \geq 0,\ \lambda_4 \cdot \frac{\partial L}{\partial \lambda_4} = 0, \quad \text{and} \tag{A33}$$

$$\frac{\partial L}{\partial \lambda_5} = r(q_1^T + q_2^T) + 2f - \underline{R} \geq 0,\ \lambda_5 \cdot \frac{\partial L}{\partial \lambda_5} = 0, \tag{A34}$$

where $D = \left\{[R-\underline{R}]^{-(\beta_1+\beta_2)}[\pi_1^T - \underline{U}_1]^{-(1-\beta_1)}[\pi_2^T - \underline{U}_2]^{-(1-\beta_2)}\right\} > 0.$

Since we concentrate on scenarios where $\pi_1^T > \underline{U}_1$, $\pi_2^T > \underline{U}_2$ and $R > \underline{R}$, which will be confirmed later, we establish $\lambda_3^* = \lambda_4^* = \lambda_5^* = 0$ from (A32)-(A34) and $\lambda_2^* = 0$ in (A31) from $r < \overline{r}$. Conversely, if $\lambda_1 > 0$, then (A30) implies $\delta^T = \delta_1 > 0$, consequently $\lambda_1^* = 0$ from (A29). This results in a contradiction. Hence, we must have $\lambda_1^* = 0$. Thus, the conditions in (A27)-(A28) become

$$\frac{\partial L}{\partial r} = \left\{(1-\beta_1-\beta_2)\frac{\partial R}{\partial r}\left[\pi_1^T - \underline{U}_1\right]\left[\pi_2^T - \underline{U}_2\right] - \frac{2\beta_1 q_1}{2+b}\left[R-\underline{R}\right]\left[\pi_2^T - \underline{U}_2\right] - \frac{2\beta_2 q_2}{2+b}\left[R-\underline{R}\right]\left[\pi_1^T - \underline{U}_1\right]\right\} \leq 0,\ r \cdot \frac{\partial L}{\partial r} = 0, \tag{A35}$$

$$\frac{\partial L}{\partial f} = \left\{2(1-\beta_1-\beta_2)\left[\pi_1^T - \underline{U}_1\right]\left[\pi_2^T - \underline{U}_2\right] - \beta_1\left[R-\underline{R}\right]\left[\pi_2^T - \underline{U}_2\right] - \beta_2\left[R-\underline{R}\right]\left[\pi_1^T - \underline{U}_1\right]\right\} \leq 0,\ f \cdot \frac{\partial L}{\partial f} = 0. \tag{A36}$$

Due to the complexity of calculations, it's not feasible to solve for all the optimal concession contracts. Nevertheless, we can demonstrate that fixed-fee contracts cannot be the optimal contract in certain scenarios as follows.

If the fixed-fee scheme is optimal, then $r^T = 0$, $f^T > 0$ and $2(1-\beta_1-\beta_2)\left[\pi_1^T - \underline{U}_1\right]\left[\pi_2^T - \underline{U}_2\right] = \beta_1\left[R-\underline{R}\right]\left[\pi_2^T - \underline{U}_2\right] + \beta_2\left[R-\underline{R}\right]\left[\pi_1^T - \underline{U}_1\right]$ according to (A36). Substituting them into (A35) results in

$$\frac{\partial L}{\partial r} = \frac{\left[R-\underline{R}\right]}{2} \cdot \frac{\partial R}{\partial r}\left[\beta_1\left(\pi_2^T - \underline{U}_2\right) + \beta_2\left(\pi_1^T - \underline{U}_1\right)\right] - \frac{2\left[R-\underline{R}\right]}{2+b}\left[\beta_1 q_1^T\left(\pi_2^T - \underline{U}_2\right) + \beta_2 q_2^T\left(\pi_1^T - \underline{U}_1\right)\right]$$

$$> \frac{1}{2}\frac{\partial R}{\partial r}\left[R-\underline{R}\right]\left[\beta_1\left(\pi_2^T-\underline{U}_2\right)+\beta_2\left(\pi_1^T-\underline{U}_1\right)\right]-\frac{2}{2+b}\left[R-\underline{R}\right]\left[\beta_1 q_1^T\left(\pi_2^T-\underline{U}_2\right)+\beta_2 q_1^T\left(\pi_1^T-\underline{U}_1\right)\right]$$

$$=\left[R-\underline{R}\right]\left[\beta_1\left(\pi_2^T-\underline{U}_2\right)+\beta_2\left(\pi_1^T-\underline{U}_1\right)\right]\left[\frac{1}{2}\cdot\frac{\partial R}{\partial r}-\frac{2q_1^T}{2+b}\right]$$

$$=\frac{\left[R-\underline{R}\right]\left[\beta_1\left(\pi_2^T-\underline{U}_2\right)+\beta_2\left(\pi_1^T-\underline{U}_1\right)\right]}{2\left(2+b\right)}\left[-2\left(q_1^T-q_2^T\right)+b\left(q_1^T+q_2^T\right)\right]. \tag{A37}$$

The first inequality above stems from $q_1^T > q_2^T$ as per Lemma 1(i). Additionally, we will have $\frac{\partial L}{\partial r} > 0$ if $-2\left(q_1^T-q_2^T\right)+b\left(q_1^T+q_2^T\right)>0$ in (A37), which will occur when

$$c_1 < c_2 < \frac{2b\left(2-b\right)a+\left(4+b^2\right)c_1}{\left(4+4b-b^2\right)}. \tag{A38}$$

However, $\frac{\partial L}{\partial r} > 0$ will contradict (A35). Consequently, the fixed-fee scheme cannot be optimal under (A38). Conversely, if (A38) is not satisfied, then the fixed-fee scheme can be optimal. These findings support Proposition 2(i).

Case 2: Suppose $\delta \in (\delta_1, \delta_2]$ with $\delta_2 \equiv \frac{a-c_1-r}{2+b} > \delta_1 > 0$. Consequently, we have $\pi_1^T = \left(q_1^T\right)^2 - f$ with $q_1^T \equiv \frac{a-c_1-r-b\delta}{2} > \delta_2$, $\pi_2^T = \frac{\delta}{2}[(2-b)(a-r)-(2-b^2)\delta+bc_1-2c_2]-f$ and $R = r(q_1^T+\delta)+2f$. Thus, the problem in (7) transforms into

$$\max_{r\ge 0, f\ge 0, \delta\ge 0} W \equiv [R-\underline{R}]^{(1-\beta_1-\beta_2)}[\pi_1^T-\underline{U}_1]^{\beta_1}[\pi_2^T-\underline{U}_2]^{\beta_2}$$

$$\text{s.t. } \delta_1 < \delta \le \delta_2,\ 0 \le r < \bar{r},\ f \ge 0,\ \pi_1^T \ge \underline{U}_1,\ \pi_2^T \ge \underline{U}_2 \text{ and } R \ge \underline{R}. \tag{A39}$$

The Lagrange function associated with the problem in (A39) is

$$L = \begin{Bmatrix} \left[R-\underline{R}\right]^{(1-\beta_1-\beta_2)}\left[\pi_1^T-\underline{U}_1\right]^{\beta_1}\left[\pi_2^T-\underline{U}_2\right]^{\beta_2}+\lambda_1\left(\delta-\delta_1\right)+\lambda_2\left(\delta_2-\delta\right)+\lambda_3\left(\bar{r}-r\right) \\ +\lambda_4\left(\pi_1^T-\underline{U}_1\right)+\lambda_5\left(\pi_2^T-\underline{U}_2\right)+\lambda_6\left[r(q_1^T+\delta)+2f-\underline{R}\right] \end{Bmatrix}$$

with the corresponding Kuhn-Tucker conditions being

$$\frac{\partial L}{\partial r} = D\cdot\left\{(1-\beta_1-\beta_2)\frac{\partial R}{\partial r}\left[\pi_1^T-\underline{U}_1\right]\left[\pi_2^T-\underline{U}_2\right]-\beta_1 q_1^T\left[\pi_2^T-\underline{U}_2\right]\left[R-\underline{R}\right]-\frac{2-b}{2}\beta_2\delta\left[\pi_1^T-\underline{U}_1\right]\left[R-\underline{R}\right]\right\}\le 0,\ r\cdot\frac{\partial L}{\partial r}=0,$$

$$\frac{\partial L}{\partial f} = D\cdot\left\{2(1-\beta_1-\beta_2)\left[\pi_1^T-\underline{U}_1\right]\left[\pi_2^T-\underline{U}_2\right]-\beta_1\left[R-\underline{R}\right]\left[\pi_2^T-\underline{U}_2\right]-\beta_2\left[R-\underline{R}\right]\left[\pi_1^T-\underline{U}_1\right]\right\}\le 0,\ f\cdot\frac{\partial L}{\partial f}=0,$$

$$\frac{\partial L}{\partial \delta} = D\cdot\begin{Bmatrix}(1-\beta_1-\beta_2)\frac{\partial R}{\partial \delta}\left[\pi_1^T-\underline{U}_1\right]\left[\pi_2^T-\underline{U}_2\right]-b\beta_1 q_1^T\left[\pi_2^T-\underline{U}_2\right]\left[R-\underline{R}\right] \\ +\frac{1}{2}\beta_2\left[(2-b)(a-r)-2(2-b^2)\delta+bc_1-2c_2\right]\left[\pi_1^T-\underline{U}_1\right]\left[R-\underline{R}\right]\end{Bmatrix}\le 0,\ \delta\cdot\frac{\partial L}{\partial \delta}=0, \tag{A40}$$

where $D=\left\{[R-\underline{R}]^{-(\beta_1+\beta_2)}[\pi_1^T-\underline{U}_1]^{-(1-\beta_1)}[\pi_2^T-\underline{U}_2]^{-(1-\beta_2)}\right\}>0$. Similar to Case 1, we have $\lambda_3^*=\lambda_4^*=\lambda_5^*=0$ and $\lambda_1^*=0$ as per $\delta>\delta_1$.

Once more, if the optimal contract is the fixed-fee scheme, then $r^T=0$, $\frac{\partial\pi_1}{\partial\delta}=-bq_1^T<0$, $\frac{\partial\pi_2}{\partial\delta}=\frac{1}{2}\left[(2-b)a-2(2-b^2)\delta+bc_1-2c_2\right]$, $\frac{\partial R}{\partial\delta}=\frac{r^T}{2}(2-b)=0$, and $\frac{\partial L}{\partial\delta}=D\left[R-\underline{R}\right]\cdot\bar{H}$ according to (A40) and $\delta>\delta_1>0$, where

$\bar{H}=-\beta_1bq_1^T\left[\pi_2^T-\underline{U}_2\right]+\frac{1}{2}\beta_2\left[(2-b)(a-r)-2(2-b^2)\delta+bc_1-2c_2\right]\left[\pi_1^T-\underline{U}_1\right]$ with

$\frac{\partial\bar{H}}{\partial\delta}=-\frac{1}{2}bq_1^T\left(\beta_1+\beta_2\right)\left[(2-b)(a-r)-2(2-b^2)\delta+bc_1-2c_2\right]+\frac{1}{2}b^2\beta_1\left(\pi_2^T-\underline{U}_2\right)-(2-b^2)\beta_2\left(\pi_1^T-\underline{U}_1\right)$ and

$$\frac{\partial^2\bar{H}}{\partial\delta^2}=\left\{\begin{array}{l}-\frac{b}{2}\left(\beta_1+\beta_2\right)\left\{-\frac{b}{2}\left[(2-b)(a-r)-2(2-b^2)\delta+bc_1-2c_2\right]-2(2-b^2)q_1^T\right\}\\+\frac{1}{4}b^2\beta_1\left[(2-b)(a-r)-2(2-b^2)\delta+bc_1-2c_2\right]+b(2-b^2)\beta_2q_1^T\end{array}\right\}.$$

Define $\hat{\delta}\equiv\frac{(2-b)(a-r)+bc_1-2c_2}{2(2-b^2)}$ and $\delta^T\in\arg\left\{\bar{H}=0\right\}$, where $\delta_2<(>)\,\hat{\delta}$ if $c_2<(>)\frac{b^2a+\left(4+2b-b^2\right)2c_1}{2\left(2+b\right)}$.

Calculations demonstrate

$$\bar{H}>(<)\,0\ \text{ at }\ \delta=\delta_1\ \text{ iff }\ \frac{1}{2}b\beta_2\delta_1\left[\left(q_1^2\right)-f-\underline{U}_1\right]>(<)\,\beta_1q_1^T\left[\left(\delta_1^2\right)-f-\underline{U}_2\right],\tag{A41}$$

$$\bar{H}>(<)\,0\ \text{ at }\ \delta=\delta_2\ \text{ iff }\left\{\begin{array}{l}\frac{\beta_2\left[b^2a+\left(4+2b-b^2\right)2c_1-2\left(2+b\right)c_2\right]}{2\left(2+b\right)}\left[\frac{\left(a-c_1\right)^2}{\left(2+b\right)^2}-f-\underline{U}_1\right]\\>(<)\frac{b\beta_1\left(a-c_1\right)}{2+b}\left[\frac{\left(a-c_1\right)\left[a+\left(1+b\right)c_1-\left(2+b\right)c_2\right]}{\left(2+b\right)^2}-f-\underline{U}_2\right]\end{array}\right\},\tag{A42}$$

$$\bar{H}=-\beta_1bq_1^T\left[\pi_2^T-\underline{U}_2\right]<0\ \text{ at }\ \delta=\hat{\delta},$$

$$\bar{H}<0\ \text{ for }\ \delta>\hat{\delta},^{7}$$

[7]If $\delta>\hat{\delta}$, then $\left[(2-b)(a-r)-2(2-b^2)\delta+bc_1-2c_2\right]<0$ and thus

$\bar{H}=-\beta_1bq_1^T\left[\pi_2^T-\underline{U}_2\right]+\frac{1}{2}\beta_2\left[(2-b)(a-r)-2(2-b^2)\delta+bc_1-2c_2\right]\left[\pi_1^T-\underline{U}_1\right]<0$ according to $\left[\pi_2^T-\underline{U}_2\right]>0$ and

$\frac{\partial \bar{H}}{\partial \delta} < 0$ at $\delta = \delta^T$,[8] and

$\frac{\partial^2 \bar{H}}{\partial \delta^2} > 0$ for $\delta \in \left(\delta_1, \hat{\delta}\right)$.

Depending on the value of c_2, there are two sub-cases as follows.

First, suppose $c_2 < \frac{b^2 a + \left(4 + 2b - b^2\right) 2c_1}{2\left(2 + b\right)}$. Then, we have $\delta_1 < \delta_2 < \hat{\delta}$. Furthermore, we have $\frac{\partial^2 \bar{H}}{\partial \delta^2} > 0$ for $\delta \in (\delta_1, \hat{\delta})$ and $\bar{H} < 0$ for $\delta \geq \hat{\delta}$, such that $\bar{H} > 0$ at $\delta = \delta_1$ if $\bar{H} > 0$ at $\delta = \delta_2$, and $\bar{H} < 0$ at $\delta = \delta_2$ if $\bar{H} < 0$ at $\delta = \delta_1$ by $\delta_1 < \delta_2$. Under these conditions, we only need to consider three cases: $\bar{H} > 0$ at $\delta = \delta_1$ and $\bar{H} < 0$ at $\delta = \delta_2$; $\bar{H} < 0$ at $\delta = \delta_1$ and $\bar{H} < 0$ at $\delta = \delta_2$; and $\bar{H} > 0$ at $\delta = \delta_1$ and $\bar{H} > 0$ at $\delta = \delta_2$. If $\bar{H} > 0$ at $\delta = \delta_1$ and $\bar{H} < 0$ at $\delta = \delta_2$, then $\delta^T \in (\delta_1, \delta_2)$ exists.[9] However, if $\bar{H} < 0$ at $\delta = \delta_1$ and $\bar{H} < 0$ at $\delta = \delta_2$, then $\bar{H} < 0$ for all $\delta \in (\delta_1, \hat{\delta})$, and consequently $\frac{\partial L}{\partial \delta} = D[R - \underline{R}] \cdot \bar{H} < 0$ for all $\delta \in (\delta_1, \hat{\delta})$ by $\frac{\partial^2 \bar{H}}{\partial \delta^2} > 0$. Therefore, if δ^T exists, we must have $\delta^T < \delta_1$. Yet, this contradicts the requirement of $\delta^T \in (\delta_1, \delta_2]$. Conversely, if $\bar{H} > 0$ at $\delta = \delta_1$ and $\bar{H} > 0$ at $\delta = \delta_2$, then $\frac{\partial L}{\partial \delta} = D[R - \underline{R}] \cdot \bar{H} > 0$ for all $\delta \in (\delta_1, \delta_2]$ by $\frac{\partial \bar{H}}{\partial \delta} < 0$ for $\delta \in (\delta_1, \delta_2]$, $\bar{H} < 0$ for $\delta > \hat{\delta}$, $\bar{H} > 0$ for $\delta \in (\delta_1, \delta_2]$, and $\frac{\partial^2 \bar{H}}{\partial \delta^2} > 0$ for $\delta \in (\delta_1, \hat{\delta})$. Thus, there exists no $\delta^T \in (\delta_1, \delta_2]$ with $\delta^T \in \arg\{\bar{H} = 0\}$.

Second, suppose $c_2 > \frac{b^2 a + \left(4 + 2b - b^2\right) 2c_1}{2\left(2 + b\right)}$. Consequently, we obtain $\delta_2 > \hat{\delta}$, thus $\bar{H} < 0$ for $\delta \in [\hat{\delta}, \delta_2)$. Under these circusmatces, if $\bar{H} > 0$ at $\delta = \delta_1$, then $\delta^T \in (\delta_1, \hat{\delta})$ exists. Conversely, if $\bar{H} < 0$ at $\delta = \delta_1$, then $\bar{H} < 0$ and $\frac{\partial L}{\partial \delta} = D[R - \underline{R}] \cdot \bar{H} < 0$ for all $\delta \in (\delta_1, \delta_2]$. Therefore, if δ^T exists, we must have $\delta^T < \delta_1$. However, this contradicts $\delta^T \in (\delta_1, \delta_2]$ again.

The aforementioned findings indicate that the fixed-fee scheme cannot be optimal if

$\left[\pi_1^T - \underline{U}_1\right] > 0$.

[8]This is because at $\delta = \delta^T$

$$\frac{\partial \bar{H}}{\partial \delta} = -\frac{1}{2} b q_1^T \left(\beta_1 + \beta_2\right) \left[(2 - b)(a - r) - 2(2 - b^2)\delta + bc_1 - 2c_2\right] - \frac{1}{4 q_1^T} \beta_2 \left(\pi_1^T - \underline{U}_1\right) \left[(4 - 2b - b^2)(a - r) - (4 - b^2)c_1 + 2bc_2\right] < 0.$$

[9]This is because $\bar{H} > 0$ at $\delta = \delta_1$, $\bar{H} < 0$ at $\delta = \delta_2$, $\frac{\partial^2 \bar{H}}{\partial \delta^2} > 0$ for $\delta \in (\delta_1, \hat{\delta})$ and $\frac{\partial \bar{H}}{\partial \delta} < 0$ at $\delta = \delta^T$.

$c_2 < \frac{b^2 a + \left(4+2b-b^2\right)2c_1}{2(2+b)}$ and $\bar{H} > 0$ at $\delta = \delta_2$ or if $c_2 > \frac{b^2 a + \left(4+2b-b^2\right)2c_1}{2(2+b)}$ and $\bar{H} < 0$ at $\delta = \delta_1$. Furthermore, $\bar{H} > 0$ at $\delta = \delta_2$ is equivalent to the condition stated in (8) according to (A42), and $\bar{H} < 0$ at $\delta = \delta_1$ is equivalent to the condition provided in (9) as per (A41). Conversely, if either (8) or (9) fails, then the fixed-fee scheme may indeed be optimal. These prove Proposition 2(ii).

Case 3: Suppose $\delta \in (\delta_2, \frac{a}{1+b})$ with $\delta_2 \equiv \frac{a-c_1-r}{2+b} > \delta_1$. Then, we have $\pi_1^T = \left[a-(1+b)\delta - c_1 - r\right]\delta - f$, $\pi_2^T = \left[a-(1+b)\delta - c_2 - r\right]\delta - f$ and $R = 2r\delta + 2f$. Thus, the problem stated in (7) transforms into

$$\max_{r\geq 0, f\geq 0, \delta\geq 0} W \equiv [R-\underline{R}]^{(1-\beta_1-\beta_2)}[\pi_1^T - \underline{U}_1]^{\beta_1}[\pi_2^T - \underline{U}_2]^{\beta_2}$$

$$\text{s.t. } \delta_2 < \delta < \frac{a}{1+b},\ 0 \leq r < \bar{r},\ f \geq 0,\ \pi_1^T \geq \underline{U}_1,\ \pi_2^T \geq \underline{U}_2 \text{ and } 2r\delta + 2f \geq \underline{R}. \qquad \text{(A43)}$$

The corresponding Lagrange function is

$$L = \begin{Bmatrix} \left[R-\underline{R}\right]^{(1-\beta_1-\beta_2)}\left[\pi_1^T - \underline{U}_1\right]^{\beta_1}\left[\pi_2^T - \underline{U}_2\right]^{\beta_2} + \lambda_1(\delta - \delta_2) + \lambda_2\left(\frac{a}{1+b} - \delta\right) + \lambda_3(\bar{r} - r) \\ + \lambda_4\left(\pi_1^T - \underline{U}_1\right) + \lambda_5\left(\pi_2^T - \underline{U}_2\right) + \lambda_6\left[2r\delta + 2f - \underline{R}\right] \end{Bmatrix},$$

where λ_1, λ_2, λ_3, λ_4, λ_5 and λ_6 are the Lagrange multipliers associated with the five inequality constraints in (A43) excluding $r \geq 0$ and $f \geq 0$. The resulting Kuhn-Tucker conditions are as follows.

$$\frac{\partial L}{\partial r} = \begin{Bmatrix} D\cdot\left\{2\delta(1-\beta_1-\beta_2)\left[\pi_1^T - \underline{U}_1\right]\left[\pi_2^T - \underline{U}_2\right] - \beta_1\delta\left[R-\underline{R}\right]\left[\pi_2^T - \underline{U}_2\right] - \beta_2\delta\left[R-\underline{R}\right]\left[\pi_1^T - \underline{U}_1\right]\right\} \\ + \frac{1}{2+b}\lambda_1 - \lambda_2 - \lambda_3 - \frac{2q_1^T}{2+b}\lambda_4 - \frac{2q_2^T}{2+b}\lambda_5 + \frac{2(a-2r)-(c_1+c_2)}{2+b}\lambda_6 \end{Bmatrix} \leq 0,\ r\cdot\frac{\partial L}{\partial r} = 0,$$

$$\frac{\partial L}{\partial f} = \left\{D\cdot\begin{Bmatrix} 2(1-\beta_1-\beta_2)\left[\pi_1^T - \underline{U}_1\right]\left[\pi_2^T - \underline{U}_2\right] \\ -\beta_1\left[R-\underline{R}\right]\left[\pi_2^T - \underline{U}_2\right] - \beta_2\left[R-\underline{R}\right]\left[\pi_1^T - \underline{U}_1\right] \end{Bmatrix} - (1-\alpha_1)\lambda_4 - (1-\alpha_2)\lambda_5 + 2\lambda_6\right\} \leq 0,\ f\cdot\frac{\partial L}{\partial f} = 0,$$

$$\frac{\partial L}{\partial \delta} = \begin{Bmatrix} D\cdot\left\{(1-\beta_1-\beta_2)\frac{\partial R}{\partial \delta}\left[\pi_1^T - \underline{U}_1\right]\left[\pi_2^T - \underline{U}_2\right] + \beta_1\frac{\partial \pi_1^T}{\partial \delta}\left[R-\underline{R}\right]\left[\pi_2^T - \underline{U}_2\right] + \beta_2\frac{\partial \pi_2^T}{\partial \delta}\left[R-\underline{R}\right]\left[\pi_1^T - \underline{U}_1\right]\right\} \\ + \lambda_1 - \lambda_2 + \left[a - 2(1+b)\delta - r - c\right]\left[\lambda_4 + \lambda_5\right] + 2r\lambda_6 \end{Bmatrix} \leq 0,\ \delta\cdot\frac{\partial L}{\partial \delta} = 0.$$

(A44)

As in Case 1, $\lambda_2^* = \lambda_3^* = \lambda_4^* = \lambda_5^* = \lambda_6^* = 0$ and $\lambda_1^* = 0$ due to the constraint of $\delta_2 < \delta < \frac{a}{1+b}$. Given that $\frac{\partial \pi_1^T}{\partial \delta} = a - 2(1+b)\delta - c_1 - r$, $\frac{\partial \pi_2^T}{\partial \delta} = a - 2(1+b)\delta - c_2 - r$ and $\frac{\partial R}{\partial \delta} = 2r$, we can infer

$\frac{\partial \pi_1^T}{\partial \delta} < 0$ and $\frac{\partial \pi_2^T}{\partial \delta} < 0$ for $\delta > \frac{a-c_1-r}{2+b} \equiv \delta_2$ due to $\frac{a-c_2-r}{2(1+b)} < \frac{a-c_1-r}{2(1+b)} < \frac{a-c_1-r}{2+b}$ by $c_1 < c_2$. Furthermore, if the fixed-fee scheme is optimal, then $r^T = 0$. Consequently, we have $\frac{\partial L}{\partial \delta} < 0$ by (A44), $\frac{\partial \pi_1^T}{\partial \delta} < 0$, $\frac{\partial \pi_2^T}{\partial \delta} < 0$ and $\frac{\partial R}{\partial \delta} = 0$ at $r^T = 0$. Therefore, the complementary slackness condition provided in (A44) implies $\delta^T = 0$, which contradicts the requirement of $\delta^T \in (\delta_2, \frac{a}{1+b})$. Hence, the fixed-fee scheme cannot be optimal in this scenario. These observations confirm Proposition 2(iii).

Proof of Proposition 3: Given that our setups differ from those of Chen and Liu (2014) and Chen et al. (2017),[10] we proceed to re-derive and re-state the results of Chen and Liu (2014) [11] as follows.

Proposition A (Chen and Liu (2014)). (i) *If* $c_2 < \hat{c}_2$ *with* $\hat{c}_2 = \frac{2(2-b)a + (2+3b)c_1}{6+b}$, *then the optimal contract is the two-part tariff scheme with* $r^{CL} = \frac{2b(2-b)a - (4+4b-b^2)c_1 + (4+b^2)c_2}{4(1+b)(2-b)}$,

$f^{CL} = \left[\frac{2(2-b)a + (2+3b)c_1 - (6+b)c_2}{4(1+b)(2-b)}\right]^2$. *At equilibrium, we have*

$q_1^{CL} = \frac{2(2-b)a - (2+b)c_1 - (2-3b)c_2}{4(1+b)(2-b)}$, $q_2^{CL} = \frac{2(2-b)a + (2+3b)c_1 - (6+b)c_2}{4(1+b)(2-b)}$,

$\pi_1^{CL} = \frac{\left[2(2-b)a + bc_1 - (4-b)c_2\right](c_2 - c_1)}{2(1+b)(2-b)^2}$, $\pi_2^{CL} = 0$, *and*

$$R^{CL} = \frac{\left[2b(2-b)a - (4+4b-b^2)c_1 + (4+b^2)c_2\right] \cdot \left[2(2-b)a + bc_1 - (4-b)c_2\right]}{8(1+b)^2(2-b)^2} + 2\left[\frac{2(2-b)a + (2+3b)c_1 - (6+b)c_2}{4(1+b)(2-b)}\right]^2.$$

(ii) *If* $\hat{c}_2 \le c_2 < \overline{c}_2$ *with* $\overline{c}_2 = \frac{(2-b)a + bc_1}{2}$, *then the optimal contract is the unit-fee scheme with* $r^{CL} = \overline{r} = \frac{(2-b)a + bc_1 - 2c_2}{(2-b)}$. *At equilibrium, we have* $q_1^{CL} = \frac{c_2 - c_1}{(2-b)}$, $q_2^{CL} = 0$, $\pi_1^{CL} = \frac{(c_2 - c_1)^2}{(2-b)^2}$, $\pi_2^{CL} = 0$, *and* $R^{CL} = \frac{(c_2 - c_1)\left[(2-b)a + bc_1 - 2c_2\right]}{(2-b)}$.

First, as shown in Case 1 of the proofs for Proposition 2 with $\delta = 0$, the corresponding Kuhn-Tucker conditions are

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[10]In Chen and Liu (2014) and Chen et al. (2017), the inverse market demand $p_i = 1 - q_i - bq_j$ is considered with an upper bound imposed on the fixed fee, i.e., $0 \le f \le \min\left\{\pi_1^{CL}, \pi_2^{CL}\right\}$. In contrast, we consider the inverse market demand $p_i = a - q_i - bq_j$ without imposing an upper bound on the fixed fee.

[11]Since the results of Chen et al. (2017) are qualitatively similar to those of Chen and Liu (2014), we focus on comparing our results with those of Chen and Liu (2014).

$\frac{\partial L}{\partial r} = D \cdot \left\{ (1-\beta_1-\beta_2)\frac{\partial R}{\partial r}\left[\pi_1^T - \underline{U}_1\right]\left[\pi_2^T - \underline{U}_2\right] - \frac{2\beta_1 q_1^T}{2+b}\left[R-\underline{R}\right]\left[\pi_2^T - \underline{U}_2\right] - \frac{2\beta_2 q_2^T}{2+b}\left[R-\underline{R}\right]\left[\pi_1^T - \underline{U}_1\right] \right\}$ and $\frac{\partial L}{\partial f} = D \cdot \left\{ 2(1-\beta_1-\beta_2)\left[\pi_1^T - \underline{U}_1\right]\left[\pi_2^T - \underline{U}_2\right] - \beta_1\left[R-\underline{R}\right]\left[\pi_2^T - \underline{U}_2\right] - \beta_2\left[R-\underline{R}\right]\left[\pi_1^T - \underline{U}_1\right] \right\}$. The corresponding Kuhn-Tucker conditions in Chen and Liu (2014) are $\frac{\partial L}{\partial r} = \frac{\partial R}{\partial r} = 0$ and $\frac{\partial L}{\partial f} = 2 > 0$. However, the Kuhn-Tucker condition for the optimal unit-fee rate in our model equals that in Chen and Liu (2014) if $\beta_1 = \beta_2 = 0$, but it equals $\frac{\partial L}{\partial r} = 0$, implying $\frac{\partial R}{\partial r} > 0$ if $\beta_1, \beta_2 > 0$. The concavity of $R(\cdot)$ implied by the second-order conditions suggests that $r^T < r^*$, and the fee revenue collected from the unit-fee scheme in our model will be less than (or equal to) that in Chen and Liu (2014) if $\beta_1, \beta_2 > 0$ (or $\beta_1 = \beta_2 = 0$). That is, $R^B \le R^{CL}$ if the unit-fee scheme is selected in both Chen and Liu (2014) and our model. However, since the two-part tariff and fixed-fee schemes can also be optimal in our model, we need to consider additional scenarios. Specifically, for small c_2, the two-part tariff scheme is optimal in Chen and Liu (2014), but the unit-fee can also be optimal in our model. In contrast, for large c_2, the unit-fee scheme is optimal in Chen and Liu (2014), while the two-part tariff and fixed-fee schemes can be optimal in our model. Therefore, it is sufficient to consider the following three cases to prove Proposition 3(i).[12]

Case 1: Suppose that the two-part tariff scheme (r^T, f^T) is selected in our model, and the two-part tariff scheme (r^{CL}, f^{CL}) given in Proposition A(i) is chosen in Chen and Liu (2014). As argued above, we have $r^T < r^{CL}$ $(r^T = r^{CL})$ and the fee revenue collected from the unit-fee part of the two-part tariff scheme in our model is less than (or equal to) that of Chen and Liu (2014) if $\beta_1, \beta_2 > 0$ (or $\beta_1 = \beta_2 = 0$).

Given any two-part tariff scheme (r, f), Lemma 1(i) implies that both operators' optimal cargo-handling amounts in Chen and Liu (2014) and our model are the same and equal to $q_1(r) = \frac{a-r}{2+b} + \frac{bc_2-2c_1}{4-b^2}$ and $q_2(r) = \frac{a-r}{2+b} + \frac{bc_1-2c_2}{4-b^2}$. Moreover, the upper bound of our fixed-fee is $f \le [q_2(r)]^2$ due to $c_2 > c_1$ and $\pi_2 = [q_2(r)]^2 - f > 0$ assumed in our model, while the optimal fixed-fee in Chen and Liu (2014) will satisfy $f = [q_2(r)]^2$. Thus, we have $r \cdot [q_1(r) + q_2(r)] + 2f < r \cdot [q_1(r) + q_2(r)] + 2[q_2(r)]^2$. Furthermore, some calculations yield

[12] If the unit-fee scheme is selected in our model and the two-part tariff scheme is chosen in Chen and Liu (2014), then we must have $R^B < R^{CL}$ because $r^T < r^{CL}$ and there is no fixed charge in our model.

$$\frac{\partial\left\{r\cdot\left[q_1(r)+q_2(r)\right]+2\left[q_2(r)\right]^2\right\}}{\partial r}=\frac{2b(2-b)a-\left(4+4b-b^2\right)c_1+\left(4+b^2\right)c_2-4(1+b)(2-b)r}{(2+b)\left(4-b^2\right)}>(<)\,0\ \text{iff}$$

$r<(>)\,r^{CL}$, where $r^{CL}=\frac{2b(2-b)a-\left(4+4b-b^2\right)c_1+\left(4+b^2\right)c_2}{4(1+b)(2-b)}$ is given in Proposition A(i). Since $r^T<r^{CL}$, we then have $R^B\equiv r^T\cdot\left(q_1^T+q_2^T\right)+2f^T<r^T\cdot\left(q_1^T+q_2^T\right)+2\left(q_2^T\right)^2<r^{CL}\cdot\left(q_1^{CL}+q_2^{CL}\right)+2\left(q_2^{CL}\right)^2\equiv R^{CL}$.

Case 2: Suppose that the two-part tariff scheme (r^T,f^T) in our model and the unit-fee scheme r^{CL} given in Proposition A(ii) of Chen and Liu (2014) are chosen. The unit-fee scheme chosen in Chen and Liu (2014) is $r^{CL}=\overline{r}$ when $c_2>\frac{2(2-b)a+(2+3b)c_1}{6+b}$ by Proposition A(ii). Under these circumstances, we have $q_2^{CL}=0$ and $r^T<r^{CL}=\overline{r}$. Thus, we have

$$R^B\equiv r^T\cdot\left(q_1^T+q_2^T\right)+2\left(q_2^T\right)^2<\left(r^T+\varepsilon\right)\cdot\left[q_1(r^T+\varepsilon)+q_2(r^T+\varepsilon)\right]+2\left[q_2(r^T+\varepsilon)\right]^2\leq\overline{r}\cdot q_1^{CL}\equiv R^{CL}$$

for some small positive number ε with $r^T+\varepsilon<\overline{r}$ because

$$\frac{\partial\left\{r\cdot\left[q_1(r)+q_2(r)\right]+2\left[q_2(r)\right]^2\right\}}{\partial r}>0\ \text{as}\ r<\overline{r} \tag{A45}$$

and $q_2(r^T+\varepsilon)\to 0$ as $(r^T+\varepsilon)$ increases to $\overline{r}$.[13]

Case 3: Suppose that the fixed-fee scheme f^T is selected in our model and the unit-fee scheme r^{CL} given in Proposition A(ii) of Chen and Liu (2014) is chosen. As in Case 2, (A45) implies

$$R^B=2f^T=0[q_1(0)+q_2(0)]+2f^T<0[q_1(0)+q_2(0)]+2\left(q_2(0)\right)^2<\varepsilon\cdot\left[q_1(\varepsilon)+q_2(\varepsilon)\right]+2\left[q_2(\varepsilon)\right]^2$$

$$<\lim_{\varepsilon\uparrow\overline{r}}\{\varepsilon\cdot\left[q_1(\varepsilon)+q_2(\varepsilon)\right]+2\left[q_2(\varepsilon)\right]^2\}=\overline{r}\cdot q_1^{CL}\equiv R^{CL}, \tag{A46}$$

where ε is a small positive number, and $q_1(\varepsilon)$ and $q_2(\varepsilon)$ are operator 1's and 2's the optimal cargo-handling amounts at $r=\varepsilon$. In (A46), the first inequality is due to $f^T<q_2(r^T)$ and $r^T=0$ is due to the fixed-fee scheme being optimal, the second inequality is implied by (A45), and the last inequality is implied by (A45) and $q_2(\overline{r})=0$.

The outcomes of Cases 1-3 then imply $R^B<R^{CL}$ as claimed by Proposition 3(i).

Second, under Assumption A1 and $c_1=c_2$, we have $r^{CL}=\frac{b(a-c)}{2(1+b)}$, $f^{CL}=\frac{(a-c)^2}{4(1+b)^2}$, $q_1^{CL}=q_2^{CL}=\frac{a-c}{2(1+b)}$, $\pi_1^{CL}=\pi_2^{CL}=0$ and $R^{CL}=\frac{(a-c)^2}{2(1+b)}$. However, given T^1 in Proposition 1, we

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[13] It is easy to check that at $r=\overline{r}=\frac{(2-b)a+bc_1-2c_2}{2-b}$, we have $\lim_{(r^T+\varepsilon)\uparrow\overline{r}}q_2(r^T+\varepsilon)=q_2(\overline{r})=0$.

have $r^T = \frac{b(a-c)}{2(1+b)}$, $f^T = \frac{[2-2(1+b)(\beta_1+\beta_2)](a-c)^2}{8(1+b)^2} - (1-\beta_1-\beta_2)\underline{U} + \frac{1}{2}(\beta_1+\beta_2)\underline{R}$, $q_1^T = q_2^T = \frac{a-c}{2(1+b)}$,

$\pi_1^T = \pi_2^T = \frac{(\beta_1+\beta_2)(a-c)^2}{4(1+b)} + (1-\beta_1-\beta_2)\underline{U} - \frac{1}{2}(\beta_1+\beta_2)\underline{R}$,

$R^T = \frac{(1-\beta_1-\beta_2)(a-c)^2}{2(1+b)} - 2(1-\beta_1-\beta_2)\underline{U} + (\beta_1+\beta_2)\underline{R}$ for $\underline{U} < \min\{\underline{U}', \underline{U}''\}$ and $\underline{R} < \frac{(a-c)^2}{2(1+b)}$,

which then imply $\Pi^{T^1} = \frac{(a-c)^2}{2(1+b)} = \Pi^{CL}$ given in Proposition A(i). Furthermore, we have $\pi_i^B \equiv \pi_i^T > \pi_i^{CL} = 0$ for $i = 1, 2$. In contrast, given U^1 in Proposition 1, we have

$r^T = \arg\left\{(1-\beta_1-\beta_2)\left[\frac{2(2+b-\alpha_1-\alpha_2)q^T - 2r}{2+b}\right][\pi^T - \underline{U}] - \frac{4(\beta_1+\beta_2)r(q^T)^2}{2+b} = 0\right\}$, $\delta^T \in \left[0, \frac{a-c-r^T}{2-b}\right]$,

$f^T = 0$, $q_1^T = q_2^T = \frac{a-r^T-c}{2+b}$, $\pi_1^T = \pi_2^T = \left(\frac{a-r^T-c}{2+b}\right)^2$, $R^T = \frac{2r^T(a-r^T-c)}{2+b}$ for

$\max\{0, \underline{U}'\} < \underline{U} < \underline{U}'''$ and $\underline{R} < \frac{b(a-c)^2[(4+3b)+b(1+b)(\beta_1+\beta_2)]}{2(1+b)^2(2+b)^2(\beta_1+\beta_2)}$, and

$\Pi^{U^1} = \frac{2(a-r^T-c)[a-c+(1+b)r^T]}{(2+b)^2} < \frac{(a-c)^2}{2(1+b)} = \Pi^{CL}$. Moreover, we have

$\pi_i^B \equiv \pi_i^T = (\frac{a-r^T-c}{2+b})^2 > \pi_i^{CL} = 0$ for $i = 1, 2$. These prove Proposition 3(ii).

It is worth noting that we do not need to consider T^2 or U^2 here, where the minimum throughput requirement is non-zero. This differs from Chen and Liu's (2014) setting, which does not account for a minimum throughput requirement.

Third, we discuss the case of Assumption A1 and $c_1 < c_2$. Here, we only compare the results of Proposition 2(i) with those of Chen and Liu (2014). The same arguments can be applied to the results of Proposition 2(ii)-(iii) when compared with those of Chen and Liu (2014). Denote $\Pi = \pi_1 + \pi_2 + R = (q_1)^2 + (q_2)^2 + r(q_1+q_2)$ as the total payoff of the port authority and the two terminal operators, where $q_i < \frac{a-r}{2+b} + \frac{bc_j - 2c_i}{4-b^2}$ for $i, j \in \{1, 2 \mid i \neq j\}$ by Lemma 1(i) with $\delta = 0$. In particular, if $q_i = q_i^{CL}$ in Chen and Liu (2014), then $\Pi = \Pi^{CL}$, and $\Pi = \Pi^B$ if $q_i = q_i^T$ in our model. The same calculations show $\frac{\partial \Pi}{\partial r} = \frac{2ba - 4(1+b)r - b(c_1+c_2)}{(2+b)^2} > (<)\, 0$ iff

$r < (>)\, \hat{r} \equiv \frac{2ba - b(c_1+c_2)}{4(1+b)}$, and $\frac{\partial^2 \Pi}{\partial r^2} = \frac{-4(1+b)}{(2+b)^2} < 0$ for all $r \geq 0$. These imply Π reaches its maximum at $r = \hat{r}$. Moreover, since $\hat{r} - r^{CL} = \frac{-2(2+b)(c_2-c_1)}{4(1+b)(2-b)} < 0$, we have $\Pi^B > \Pi^{CL}$ if

$\hat{r} < r^T < r^{CL}$; and $\prod^B < \prod^{CL}$ if $\hat{r} < r^{CL} < r^T$. However, the relative sizes of $\prod^B$ and $\prod^{CL}$ are uncertain if $r^T < \hat{r} < r^{CL}$. Thus, it is enough to determine the relative size of r^T and r^{CL}. According to the values of $\frac{\partial L}{\partial f}$ and $\frac{\partial L}{\partial r}$, there are two sub-cases as follows.

(i) Suppose $\frac{\partial L}{\partial f} = 0$ and $\frac{\partial L}{\partial r} = 0$. That is, the two-part tariff scheme is selected in our model. We then have $2(1-\beta_1-\beta_2)\left[\pi_1^T - \underline{U}_1\right]\left[\pi_2^T - \underline{U}_2\right] = \beta_1\left[R-\underline{R}\right]\left[\pi_2^T - \underline{U}_2\right] + \beta_2\left[R-\underline{R}\right]\left[\pi_1^T - \underline{U}_1\right]$ by $\frac{\partial L}{\partial f} = 0$. Substituting it into $\frac{\partial L}{\partial r}$ yields $\frac{\partial L}{\partial r} = D \cdot \frac{\left[R-\underline{R}\right]}{2} \cdot K$, where

$K \equiv \left[\frac{\partial R}{\partial r} - \frac{4q_1^T}{2+b}\right]\beta_1\left[\pi_2^T - \underline{U}_2\right] + \left[\frac{\partial R}{\partial r} - \frac{4q_2^T}{2+b}\right]\beta_2\left[\pi_1^T - \underline{U}_1\right]$. Since $\frac{\partial L}{\partial r} = 0$ at $r = r^T$ and $\frac{\partial L}{\partial r} < 0$ at $r = r^{CL}$ by calculations, we have $r^T < r^{CL}$ based on the concavity of $L(\cdot)$. Conversely, we have $\prod\left(r=0\right) - \prod\left(r = r^{CL}\right) = \frac{-br^{CL}\left[2a-\left(c_1+c_2\right)\right]}{\left(2+b\right)^2} < 0$ at $r = r^{CL}$. Therefore, there exists $\hat{\hat{r}}$ with $0 < \hat{\hat{r}} < \hat{r}$ satisfying $\prod\left(r = \hat{\hat{r}}\right) = \prod\left(r = r^{CL}\right)$ due to the concavity of $\prod(\cdot)$. These then imply $\prod^B > (<) \prod^{CL}$ if $r^T > (<) \hat{\hat{r}}$. Moreover, further calculations yield

$$\hat{\hat{r}} = \frac{2b\left(2-b\right)a + \left(4+b^2\right)c_1 - \left(4+4b-b^2\right)c_2}{4\left(1+b\right)\left(2-b\right)} > 0 \text{ if } c_2 < \frac{2b\left(2-b\right)a + \left(4+b^2\right)c_1}{\left(4+4b-b^2\right)};$$

$$K = \frac{4\beta_2\left(c_2-c_1\right)}{4-b^2}\left\{\left[\frac{2(2-b)a-(6+b)c_1+(2+3b)c_2}{4(1+b)(2-b)}\right]^2 - f^T - \underline{U}_1\right\} > 0 \text{ at } r = \hat{\hat{r}};$$

$$\frac{\partial K}{\partial r} = \left\{-\frac{4(1+b)}{(2+b)^2}\left\{\begin{array}{l}\beta_1\left[\left[\frac{2(2-b)a-(2-3b)c_1-(2+b)c_2}{4(1+b)(2-b)}\right]^2 - f^T - \underline{U}_2\right] \\ +\beta_2\left[\left[\frac{2(2-b)a-(6+b)c_1+(2+3b)c_2}{4(1+b)(2-b)}\right]^2 - f^T - \underline{U}_1\right]\end{array}\right\} - \frac{8\beta_2(c_2-c_1)\left[2(2-b)a-(6+b)c_1+(2+3b)c_2\right]}{4(1+b)\left(4-b^2\right)^2}\right\} < 0$$

at $r = \hat{\hat{r}}$; and

$$\frac{\partial L}{\partial r} = D \cdot \frac{(R-\underline{R})}{2} \cdot K > 0 \text{ at } r = \hat{\hat{r}}.$$

These findings imply $r^{CL} > r^T > \hat{\hat{r}}$ due to the concavity of $L(\cdot)$, $\frac{\partial L}{\partial r} = 0$ at $r = r^T$, and $r^T < r^{CL}$. Furthermore, under the unit-fee scheme r and Lemma 1(i) with $\delta = 0$ and $c_1 = c_2$, we have both operators' optimal cargo-handling amounts being $q_1(r) = q_2(r) = \frac{a-r-c}{2+b}$ with

$\frac{\partial q_1(r)}{\partial r} = \frac{\partial q_2(r)}{\partial r} = \frac{-1}{2+b} < 0$ and $\frac{\partial\left\{\left[q_1(r)\right]^2 - \left[q_2(r)\right]^2\right\}}{\partial r} = \frac{-2\left(c_2-c_1\right)}{4-b^2} < 0$. These suggest $q_1^T > q_1^{CL}$,

$q_2^T > q_2^{CL}$, $\pi_1^T = \left(q_1^T\right)^2 - f^T > \left(q_1^T\right)^2 - \left(q_2^T\right)^2 > \left(q_1^{CL}\right)^2 - \left(q_2^{CL}\right)^2 = \left(q_1^{CL}\right)^2 - f^{CL} = \pi_1^{CL}$ by $f^T < \left(q_2^T\right)^2$ and $f^{CL} = \left(q_2^{CL}\right)^2$, and $\pi_2^T > \pi_2^{Cl} = 0$ by $\pi_2^T = \left(q_2^T\right)^2 - f^T > 0$.

In summary, if the two-part tariff scheme is offered, then we must have $\prod^B > \prod^{CL}$ and $\pi_i^B \equiv \pi_i^T > \pi_i^{CL}$ for $i = 1, 2$. These validate the first part of Proposition 3(iii).

(ii) Suppose $\frac{\partial L}{\partial f} < 0$ and $\frac{\partial L}{\partial r} = 0$. If $\frac{\partial L}{\partial f} < 0$, then $f^T = 0$, and hence the unit-fee scheme is selected in our model. At $f^T = 0$, $\frac{\partial L}{\partial r} = 0$ implies $\frac{\partial L}{\partial r} = D \cdot J$, where $J = (1-\beta_1-\beta_2)\frac{\partial R}{\partial r}\left[\pi_1^T - \underline{U}_1\right]\left[\pi_2^T - \underline{U}_2\right] - \frac{2\beta_1 q_1^T}{2+b}\left[R - \underline{R}\right]\left[\pi_2^T - \underline{U}_2\right] - \frac{2\beta_2 q_2^T}{2+b}\left[R - \underline{R}\right]\left[\pi_1^T - \underline{U}_1\right]$. If $J < 0$ at $r = \hat{\hat{r}}$, then $\frac{\partial L}{\partial r} < 0$ at $r = \hat{\hat{r}}$, and hence $r^T < \hat{\hat{r}}$ by $\frac{\partial L}{\partial r} = 0$ at $r = r^T$, which in turn implies $\prod^B < \prod^{CL}$ as argued in part (i). Some calculations yield $J < 0$ at $r = \hat{\hat{r}}$ if $c_2 < \frac{2b(2-b)a + \left(4+b^2\right)c_1}{\left(4+4b-b^2\right)}$ and $(b, c_1, c_2, \beta_1, \beta_2, \underline{U}_1, \underline{U}_2, \underline{R})$ satisfy $G_1 < 0$, where

$$G_1 = \left\{ \begin{array}{l} \frac{2(2-b)a-(6+b)c_1+(2+3b)c_2}{2(1+b)\left(4-b^2\right)} \left\{ \left[\frac{2(2-b)a-(2-3b)c_1-(2+b)c_2}{4(1+b)(2-b)} \right]^2 - \underline{U}_2 \right\} \left\{ \begin{array}{l} 2(1-\beta_1-\beta_2)\left[\left[\frac{2(2-b)a-(6+b)c_1+(2+3b)c_2}{4(1+b)(2-b)} \right]^2 - \underline{U}_1 \right] \\ -\beta_1 \left[\frac{\left[2b(2-b)a+\left(4+b^2\right)c_1-\left(4+4b-b^2\right)c_2\right]\left[2(2-b)a-(4-b)c_1+bc_2\right]}{8(1+b)^2(2-b)^2} - \underline{R} \right] \end{array} \right\} \\ -\frac{\beta_2\left[2(2-b)a-(2-3b)c_1-(2+b)c_2\right]}{2(1+b)\left(4-b^2\right)} \left[\frac{\left[2b(2-b)a+\left(4+b^2\right)c_1-\left(4+4b-b^2\right)c_2\right]\left[2(2-b)a-(4-b)c_1+bc_2\right]}{8(1+b)^2(2-b)^2} - \underline{R} \right] \left[\left[\frac{2(2-b)a-(6+b)c_1+(2+3b)c_2}{4(1+b)(2-b)} \right]^2 - \underline{U}_1 \right] \end{array} \right\}.$$

And $J > 0$ at $r = r^*$ if $(b, c_1, c_2, \beta_1, \beta_2, \underline{U}_1, \underline{U}_2, \underline{R})$ satisfy $G_2 > 0$, where

$$G_2 = \left\{ \begin{array}{l} \frac{2(2-b)a-(6+b)c_2+(2+3b)c_1}{2(1+b)\left(4-b^2\right)} \left\{ \left[\frac{2(2-b)a-(2-3b)c_2-(2+b)c_1}{4(1+b)(2-b)} \right]^2 - \underline{U}_1 \right\} \left\{ \begin{array}{l} 2(1-\beta_1-\beta_2)\left[\left[\frac{2(2-b)a-(6+b)c_2+(2+3b)c_1}{4(1+b)(2-b)} \right]^2 - \underline{U}_2 \right] \\ -\beta_1 \left[\frac{\left[2b(2-b)a+\left(4+b^2\right)c_2-\left(4+4b-b^2\right)c_1\right]\left[2(2-b)a-(4-b)c_2+bc_1\right]}{8(1+b)^2(2-b)^2} - \underline{R} \right] \end{array} \right\} \\ -\frac{\beta_2\left[2(2-b)a-(2-3b)c_2-(2+b)c_1\right]}{2(1+b)\left(4-b^2\right)} \left[\frac{\left[2b(2-b)a+\left(4+b^2\right)c_2-\left(4+4b-b^2\right)c_1\right]\left[2(2-b)a-(4-b)c_2+bc_1\right]}{8(1+b)^2(2-b)^2} - \underline{R} \right] \left[\left[\frac{2(2-b)a-(6+b)c_2+(2+3b)c_1}{4(1+b)(2-b)} \right]^2 - \underline{U}_1 \right] \end{array} \right\}.$$

We need to verify the consistency between the conditions on c_2. Note that the two-part tariff scheme is optimal in Chen and Liu (2014) if $c_2 < \frac{2a(2-b)+(2+3b)c_1}{6+b}$. Through calculations, it is found that $\frac{2(2-b)a+(2+3b)c_1}{6+b} - \frac{2b(2-b)a+\left(4+b^2\right)c_1}{\left(4+4b-b^2\right)} = \frac{4(1-b)\left(4-b^2\right)(a-c_1)}{(6+b)\left(4+4b-b^2\right)} > 0$. Consequently, we will have $\prod^B < \prod^{CL}$ if $c_2 < \frac{2b(2-b)a+\left(4+b^2\right)c_1}{\left(4+4b-b^2\right)}$ and the unit-fee scheme is selected in our model. As in part (i), we have $r^T < r^{CL}$ and hence $\pi_1^B \equiv \pi_1^T = \left(q_1^T\right)^2 > \left(q_1^{CL}\right)^2 > \left(q_1^{CL}\right)^2 - f^{CL} = \pi_1^{CL}$ and $\pi_2^B \equiv \pi_2^T = \left(q_2^T\right)^2 > 0 = \pi_2^{CL}$. These then substantiate the second part of Proposition 3(iii).

(iii) Suppose $\frac{\partial L}{\partial f} = 0$ and $\frac{\partial L}{\partial r} < 0$. If $\frac{\partial L}{\partial r} < 0$, then $r^T = 0$ and hence the fixed-fee scheme is selected in our model. In addition, we have $\Pi^B > \Pi^{CL}$ if $\hat{\hat{r}} < r^T < \hat{r}$, $\Pi^B < \Pi^{CL}$ if $0 \equiv r^T < \hat{\hat{r}} < \hat{r}$, and $\hat{\hat{r}} > (\leq) 0$ if $c_2 < (\geq) \frac{2b(2-b)a + (4+b^2)c_1}{(4+4b-b^2)}$. These imply $\Pi^B < \Pi^{CL}$ if $c_2 < \frac{2b(2-b)a + (4+b^2)c_1}{(4+4b-b^2)}$; and $\Pi^B > \Pi^{CL}$ if $\frac{2b(2-b)a + (4+b^2)c_1}{(4+4b-b^2)} < c_2 < \frac{2(2-b)a + (2+3b)c_1}{6+b}$. As in Case 1 of the proofs for Proposition 2, the fixed-fee scheme can be optimal if (A38) fails. Thus, $r^T = 0 < r^{CL}$ if $c_2 \geq \frac{2b(2-b)a + (4+b^2)c_1}{(4+4b-b^2)}$, which in turn implies

$\pi_1^B \equiv \pi_1^T = \left(q_1^T\right)^2 - f^T > \left(q_1^T\right)^2 - \left(q_2^T\right)^2 > \left(q_1^{CL}\right)^2 - \left(q_2^{CL}\right)^2 = \left(q_1^{CL}\right)^2 - f^{CL} = \pi_1^{CL}$ and

$\pi_2^B \equiv \pi_2^T = \left(q_2^T\right)^2 - f^T > 0 = \pi_2^{CL}$. These then substantiate the third part of Proposition 3(iii).

Proof of Proposition 4: We first restate the results of Liu et al. (2018) within the context of our setup as follows.

Proposition B (Liu et al. (2018)). (i) *If* $c_2 \in \left(c_1, \ddot{c}_2\right]$ *with*

$$\ddot{c}_2 = \frac{\left[(2+b)(3-2b)a + (1+b)(2-b)c_1\right] - \sqrt{2(1+b)(3-2b)(2-b^2)(a-c_1)^2}}{(8-3b^2)}$$

, then the port authority's best choice is the unit-fee scheme. The optimal contract and minimum throughput guarantee are $r^L = \frac{(a-c_2)}{2}$ *and* $\delta^L = \frac{(a-c_2)}{2(1+b)}$. *At equilibrium, we have* $q_i^L = \delta^L = \frac{a-c_2}{2(1+b)}$ *for* $i = 1,2$,

$\pi_1^L = \frac{(c_2-c_1)(a-c_2)}{2(1+b)^2}$, $\pi_2^L = 0$, *and* $R^L = \frac{(a-c_2)^2}{2(1+b)}$.

(ii) *If* $c_2 \in \left(\ddot{c}_2, \tilde{c}_2\right)$ *with* $\tilde{c}_2 \equiv \frac{(2-b)(3-2b)a + (2+3b-3b^2)c_1}{(8-4b-b^2)}$, *then the unit-fee scheme is the port authority's best choice. The optimal contract and minimum throughput guarantee are*

$r^L = \frac{1}{2(3-2b)}\left[(3-2b)a-(1-b)c_1-(2-b)c_2\right]$ *and* $\delta^L = \frac{(2-b)(a-r^u)+bc_1-2c_2}{(2-b^2)}$. *At equilibrium, we have* $q_1^L = \frac{1}{2}(a-b\delta^L-c_1-r) > \delta^L$, $q_2^L = \delta^L$, $\pi_1^L = (q_1^L)^2$, $\pi_2^L = 0$, *and* $R^L = \frac{\left[(3-2b)a-(1-b)c_1-(2-b)c_2\right]^2}{4(3-2b)(2-b^2)}$.

(iii) *If* $c_2 \in [\tilde{c}_2, \bar{c}_2)$ *with* $\bar{c}_2 = \frac{(2-b)a+bc_1}{2}$, *then the port authority's best choice is the unit-fee scheme. The optimal contract and minimum throughput guarantee are* $r^L = \bar{r} \equiv \frac{(2-b)a+bc_1-2c_2}{2-b}$ *and* $\delta^L = 0$. *At equilibrium, we have* $q_1^L = \frac{c_2-c_1}{2-b}$, $q_2^L = 0$, $\pi_1^L = \frac{(c_2-c_1)^2}{(2-b)^2}$, $\pi_2^L = 0$, *and* $\bar{R}^L = \frac{(c_2-c_1)\left[(2-b)a+bc_1-2c_2\right]}{(2-b)^2}$.

First, we can demonstrate Proposition 4(i) using arguments similar to those of Proposition 3(i), except that we will adopt the Kuhn-Tucker conditions as in Case 2 of Proposition 2.

Second, under Assumption A1 and $c_1 = c_2$, Proposition B implies $r^L = \frac{(a-c)}{2}$, $\delta^L = \frac{(a-c)}{2(1+b)}$, $q_i^L = \delta^L = \frac{a-c}{2(1+b)}$ for $i = 1,2$, $\pi_1^L = \pi_2^L = 0$, and $R^L = \frac{(a-c)^2}{2(1+b)}$, which then correspond to $\prod^L \equiv \pi_1^L + \pi_2^L + R^L = \prod^{CL}$ obtained in Chen and Liu (2014). Moreover, as proved in Proposition 1, the port authority's equilibrium fee revenue, operators' equilibrium profits, and their total payoff under the contracts T^1, T^2 and U^2 are the same. Thus, by following the same arguments as those in Proposition 3(ii), we can conclude $\prod^B < \prod^L$ for $\beta_1, \beta_2 > 0$ if U^1 is adopted, $\prod^B = \prod^L$ if T^1, T^2 or U^2 is adopted, and $\pi_i^B > \pi_i^L = 0$ for $i = 1,2$ regardless of the optimal contract types. These prove Proposition 4(ii).

Third, under Assumption A1 and $c_1 < c_2$, we will compare the results of Proposition 2(i) with those of Proposition B(i)-(ii).[14] In this context, there are two subcases as follows.

Case a: Suppose $q_1^L > \delta$ and $q_2^L = \delta$ (or $q_1^T > \delta$ and $q_2^T = \delta$) are given in Lemma 1(ii).[15] In this situation, the total payoff of the port authority and the two terminal operators in our model is $\prod(r,\delta) \equiv \pi_1 + \pi_2 + R = (q_1)^2 + \frac{\delta}{2}[(2-b)(a-r)-(2-b^2)\delta+bc_1-2c_2] + r(q_1+\delta)$, where $q_1 = \frac{a-c_1-r-b\delta}{2}$. In particular, we have $\Pi^B = \prod(r^T,\delta^T)$ and $\Pi^L = \prod(r^L,\delta^L)$, where (r^T, δ^T) is the

[14]In Proposition B(iii), $\delta^L = 0$ and $q_2^L = 0$, which are not particularly interesting. Therefore, we focus on the results of Proposition B(i)-(ii).

[15]Note that the operators' optimal cargo-handling amounts given in Lemma 1 are the same as those in Liu et al. (2018).

optimal contract derived in Proposition 1 or 2, and $(r^L,\ \delta^L)$ is the optimal contract derived in Proposition B(ii) for $c_2 \in (\ddot{c}_2, \tilde{c}_2)$.

By some calculations, we have $\frac{\partial \Pi}{\partial \delta} = \frac{2(1-b)a + br + 2bc_1 - 2c_2 - (4-3b^2)\delta}{2}$ and $\frac{\partial \Pi}{\partial r} = \frac{-r + b\delta}{2}$. Letting $\frac{\partial \Pi}{\partial r} = \frac{\partial \Pi}{\partial \delta} = 0$ yields $\breve{r} \equiv \frac{b\left[(1-b)a + bc_1 - c_2\right]}{2(1-b^2)}$ and $\breve{\delta} \equiv \frac{(1-b)a + bc_1 - c_2}{2(1-b^2)}$, which produce the maximum of $\Pi(r,\delta)$. Given $\{\breve{r}, \breve{\delta}\}$, we have

$(\breve{r} - r^L) = \frac{-(3-5b+2b^2)(a-c_2)}{2(1-b^2)(3-2b)} < 0$ and

$(\delta^L - \breve{\delta}) = \frac{(3b-5b^2+2b^3)a + (2-3b-b^2+b^4)c_1 - (2-6b^2+2b^3+b^4)c_2}{2(3-2b)(1-b^2)(2-b^2)}$. Since $\Pi(r,\delta)$ is a multi-variable function of r and δ, we cannot directly compare Π^B and Π^L.[16] Thus, we will first find the anchor (or reference) point r with $r \neq r^L$ satisfying the condition of $\Pi(r,\delta^L) = \Pi^L$ given $\delta = \delta^L$, or the anchor (or reference) point δ with $\delta \neq \delta^L$ satisfying the condition of $\Pi(r^L,\delta) = \Pi^L$ given $r = r^L$. Note that these anchor points will exist due to the concavity of $\Pi(r,\delta)$. Afterwards, we will compare these anchor points with $(r^T,\ \delta^T)$, and then we can determine the relative sizes of Π^B and Π^L.

Since our optimal contracts include the two-part tariff, unit-fee and fixed-fee scheme as shown in Proposition 2, we will compare our optimal two-part tariff contract with that of Liu et al. (2018). The same arguments can be applied to the other optimal contracts to reach similar conclusions.

First, we try to find the anchor point of r given $\delta = \delta^L$. Since $\breve{r} < r^L$, there exists $\breve{\breve{r}}$ at $\delta = \delta^L$ with $\breve{\breve{r}} = \frac{(-6+16b-11b^2+2b^3)a + (2+2b+5b^2-5b^3)c_1 + (4-18b+6b^2+3b^3)c_2}{(12-8b-6b^2+4b^3)} < 0$ satisfying $\breve{\breve{r}} < \breve{r} < r^L$ and solving $\Pi(r,\delta^L) = \Pi^L$. This implies that at $\delta^T = \delta^L$, we have $\Pi^B < \Pi^L$ if $r^T < \breve{\breve{r}}$ due to $\Pi(r^T,\delta^L) < \Pi(\breve{\breve{r}},\delta^L) = \Pi(r^L,\delta^L)$ by $\frac{\partial \Pi}{\partial r} > 0$ for $r < \breve{r}$. However, since $\breve{\breve{r}} < 0$, we have $\breve{\breve{r}} < 0 < r^T$ and then $\Pi(r^T,\delta^L) > \Pi(\breve{\breve{r}},\delta^L) = \Pi(r^L,\delta^L)$ by $\Pi^B > \Pi^L$. Thus, this anchor point $(\breve{\breve{r}},\delta^L)$ is meaningless due to $\breve{\breve{r}} < 0$.

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[16] First, since (r^T,δ^T) in our model cannot be analytically solved in some cases, it is not possible to directly compare $\Pi(r^T,\ \delta^T)$ and $\Pi(r^L,\ \delta^L)$ by substituting $(r^T,\ \delta^T)$ and $(r^L,\ \delta^L)$ into $\Pi(r,\delta)$. Second, even though we can substitute $(r^L,\ \delta^L)$ into the first-order conditions for deriving $(r^T,\ \delta^T)$, we still cannot determine the relative sizes of $\Pi(r^T,\ \delta^T)$ and $\Pi(r^L,\ \delta^L)$ because $\Pi(r,\delta)$ is a mutli-variable function of r and δ.

Second, we try to find the anchor point of δ given $r = r^L$. We compare $\prod^B$ with $\prod^L$ at $(r = r^L, \delta = \breve{\breve{\delta}})$, where $\breve{\breve{\delta}}$ with $\breve{\breve{\delta}} < \breve{\delta}$ satisfies $\prod(r^L, \delta) = \Pi^L$. This then implies $\prod^B < \prod^L$ if $\delta^T < \breve{\breve{\delta}}$, and $\prod^B > \prod^L$ if $\delta^T > \breve{\breve{\delta}}$. By some calculations, we have

$$\breve{\breve{\delta}} = \frac{\left(24 - 40b + 10b^2 + 13b^3 - 6b^4\right)a - \left(8 - 32b + 10b^2 + 13b^3 - 5b^4\right)c_1 - \left(16 - 8b - b^4\right)c_2}{2\left(3 - 2b\right)\left(2 - b^2\right)\left(4 - 3b^2\right)} > 0 \text{ if } b < 0.6309,$$

$$\delta^L - \breve{\breve{\delta}} = \frac{2\left(3b - 5b^2 + 2b^3\right)a + 2\left(4 - 5b - 2b^2 + b^3 + b^4\right)c_1 - 2\left(4 - 2b - 7b^2 + 3b^3 + b^4\right)c_2}{\left(3 - 2b\right)\left(2 - b^2\right)\left(4 - 3b^2\right)} = 2\left(\delta^L - \breve{\delta}(r^L)\right) > 0$$

if $0.4343 < b < 0.6309$ and $\ddot{c}_2 < c_2 < c_2^d$,[17] or if $b > 0.6309$ for $\ddot{c}_2 < c_2 < \tilde{c}_2$, where

$$c_2^d = \frac{\left(3b - 5b^2 + 2b^3\right)a + \left(4 - 5b - 2b^2 + b^3 + b^4\right)c_1}{\left(4 - 2b - 7b^2 + 3b^3 + b^4\right)} \text{ and}$$

$$\breve{\delta}(r^L) = \frac{\left(12 - 17b + 6b^2\right)a + \left(11b - 7b^2\right)c_1 - \left(12 - 6b - b^2\right)c_2}{2\left(3 - 2b\right)\left(4 - 3b^2\right)}.$$

Thus, if $\frac{\partial L}{\partial r} \geq 0$, $\frac{\partial L}{\partial \delta} < 0$, $\frac{\partial^2 L}{\partial r^2} < 0$ and $\frac{\partial^2 L}{\partial \delta^2} < 0$ at $(r = r^L,\ \delta = \breve{\breve{\delta}})$, then $r^T > r^L$ and $\delta^T < \breve{\breve{\delta}}$, and hence $\prod^B < \prod^L$, where the Lagrange function $L(\cdot)$ is defined in Case 1 of the proofs for Proposition 2.[18] In contrast, if $\frac{\partial L}{\partial r} \leq 0$ and $\frac{\partial L}{\partial \delta} > 0$ at $(r = r^L,\ \delta = \breve{\breve{\delta}})$, then $r^T < r^L$ and $\delta^T > \breve{\breve{\delta}}$, and hence $\prod^B > \prod^L$.

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[17]Because $\pi_2^L = \frac{\delta}{2}[(2-b)(a-r) - (2-b^2)\delta + bc_1 - 2c_2]$ and $\pi_2^L \leq 0$ if $\delta \geq \delta^L = \frac{(2-b)\left(a - r^L\right) + bc_1 - 2c_2}{\left(2 - b^2\right)}$, any δ greater than the value of δ^L will result in $\pi_2^L < 0$, which is meaningless. Therefore, for the relationship between δ^L and $\breve{\delta}(r^L)$, we only consider the case where $\delta^L > \breve{\delta}(r^L)$.

[18]The second-order conditions for the solutions derived from $\frac{\partial L}{\partial r} = \frac{\partial L}{\partial f} = \frac{\partial L}{\partial \delta} = 0$ is the negative semidefinite Hessian matrix of $L(r,\ f, \delta)$, $\begin{bmatrix} \frac{\partial^2 L}{\partial r^2} & \frac{\partial^2 L}{\partial f \partial r} & \frac{\partial^2 L}{\partial \delta \partial r} \\ \frac{\partial^2 L}{\partial r \partial f} & \frac{\partial^2 L}{\partial f^2} & \frac{\partial^2 L}{\partial \delta \partial f} \\ \frac{\partial^2 L}{\partial r \partial \delta} & \frac{\partial^2 L}{\partial f \partial \delta} & \frac{\partial^2 L}{\partial \delta^2} \end{bmatrix}$. Specifically, we have $\text{sign}(\frac{\partial^2 L}{\partial r^2}) = \text{sign}(\frac{\partial W}{\partial r})$ and $\text{sign}(\frac{\partial^2 L}{\partial \delta^2}) = \text{sign}(\frac{\partial S}{\partial \delta})$ with $\frac{\partial W}{\partial r} = \frac{-\beta_1\left[-(a-c_1) + (2+b)\delta\right]\left\{(2-b)\delta\left[(a-c_1) - 2r - (2-b)\delta\right] - 4\left[\pi_2^T - \underline{U}_2\right]\right\}}{4\left[(a-c_1) - 2r - (2-b)\delta\right]} - \frac{1}{2}\beta_2 q_1^T\left[(a-c_1) - 2r - (2-b)\delta\right]$ and

$$\frac{\partial S}{\partial \delta} = \left\{ \begin{array}{l} \frac{-\beta_2\left[\pi_2^T - \underline{U}_2\right]\left\{-4b\left(4 - 2b - b^2\right)a + \left(16 + 8b - 12b^2 - 2b^3\right)r + 4b\left(4 - b^2\right)c_1 - 8b^2 c_2\right\}}{\left[-2b(a-c_1) + (2+b)r + 2b^2\delta\right]} \\ -4\left(2 - b^2\right)\beta_2\left[\pi_1^T - \underline{U}_1\right] - bq_1^T\beta_2\left[2(2-b)\text{a} - (2-b)r - 4\left(2 - b^2\right)\delta + 2bc_1 - 4c_2\right] \end{array} \right\}.$$

Thus, $\frac{\partial W}{\partial r} < 0$, $\frac{\partial S}{\partial \delta} < 0$, $\frac{\partial^2 L}{\partial f^2} < 0$ and sufficiently large $|\frac{\partial W}{\partial r}|$, $|\frac{\partial S}{\partial \delta}|$ and $|\frac{\partial^2 L}{\partial f^2}|$ are assumed to guarantee that the second-order conditions hold.

In the following, we focus on the interior solution (i.e., the two-part tariff scheme) derived by $\frac{\partial L}{\partial r}=\frac{\partial L}{\partial f}=\frac{\partial L}{\partial \delta}=0$, and find the conditions leading to $\prod^B<\prod^L$.[19] We have

$2(1-\beta_1-\beta_2)\left[\pi_1^T-\underline{U}_1\right]\left[\pi_2^T-\underline{U}_2\right]=\beta_1\left[R-\underline{R}\right]\left[\pi_2^T-\underline{U}_2\right]+\beta_2\left[R-\underline{R}\right]\left[\pi_1^T-\underline{U}_1\right]$ by $\frac{\partial L}{\partial f}=0$.

Substituting it into $\frac{\partial L}{\partial r}$ yields $\frac{\partial L}{\partial r}=D\cdot\frac{(R-\underline{R})}{2}\cdot W$, where

$W=\beta_1\left[\frac{\partial R}{\partial r}-2q_1^T\right]\left[\pi_2^T-\underline{U}_2\right]+\beta_2\left[\frac{\partial R}{\partial r}-(2-b)\delta\right]\left[\pi_1^T-\underline{U}_1\right]$ with

$\frac{\partial W}{\partial r}=-\frac{2-b}{4}\beta_1\delta\left[-(a-c_1)+(2+b)\delta\right]+\frac{1}{2}\beta_2\left\{-2\left[\pi_1^T-\underline{U}_1\right]-q_1^T\left[(a-c_1)-2r-(2-b)\delta\right]\right\}$. Substituting it into $\frac{\partial L}{\partial \delta}$ yields $\frac{\partial L}{\partial \delta}=\frac{DS\left[R-\underline{R}\right]}{4}$, where

$S=\beta_1\left[-2b(a-c_1)+(2+b)r+2b^2\delta\right]\left[\pi_2^T-\underline{U}_2\right]+\beta_2\left[2(2-b)\mathrm{a}-(2-b)r-4\left(2-b^2\right)\delta+2bc_1-4c_2\right]\left[\pi_1^T-\underline{U}_1\right]$

with $\frac{\partial S}{\partial r}=\left\{\begin{array}{l}\beta_1\left\{(2+b)\left[\pi_2^T-\underline{U}_2\right]-\frac{(2-b)\delta}{2}\left[-2b(a-c_1)+(2+b)r+2b^2\delta\right]\right\}\\+\beta_2\left\{-(2-b)\left[\pi_1^T-\underline{U}_1\right]-q_1^T\left[2(2-b)\mathrm{a}-(2-b)r-4\left(2-b^2\right)\delta+2bc_1-4c_2\right]\right\}\end{array}\right\}$ and

$$\frac{\partial S}{\partial \delta}=\left\{\begin{array}{l}2b^2\beta_1\left[\pi_2^T-\underline{U}_2\right]+\frac{1}{2}\beta_1\left[-2b(a-c_1)+(2+b)r+2b^2\delta\right]\left[(2-b)(a-r)-2(2-b^2)\delta+bc_1-2c_2\right]\\-4\left(2-b^2\right)\beta_2\left[\pi_1^T-\underline{U}_1\right]-bq_1^T\beta_2\left[2(2-b)\mathrm{a}-(2-b)r-4\left(2-b^2\right)\delta+2bc_1-4c_2\right]\end{array}\right\}.$$

Moreover, at $(r=r^L,\ \delta=\breve{\delta})$, we have $W=A\beta_1\left[\pi_2^T-\underline{U}_2\right]+B\beta_2\left[\pi_1^T-\underline{U}_1\right]$ with

$$A=\frac{\left[-\left(24b-40b^2+4b^3+17b^4-6b^5\right)a+\left(32+24b-48b^2+4b^3+15b^4-7b^5\right)c_1-\left(32-8b^2-2b^4-b^5\right)c_2\right]}{4(3-2b)\left(2-b^2\right)\left(4-3b^2\right)}$$ and

$$B=\frac{(2-b)\left[-\left(24-40b+10b^2+13b^3-6b^4\right)a-\left(8+32b-30b^2-13b^3+11b^4\right)c_1+\left(32-8b-20b^2+5b^4\right)c_2\right]}{4(3-2b)\left(2-b^2\right)\left(4-3b^2\right)}.$$

Some calculations show $A<\frac{-2\left(4+10b-10b^2-3b^3+3b^4\right)(a-c_1)}{\left(4-3b^2\right)\left(10-b-4b^2\right)}<0$ by

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[19]As shown in Case 3 of the proof of Proposition 2, the signs of $\frac{\partial L}{\partial r}$ and $\frac{\partial L}{\partial f}$ are the same, and hence $\frac{\partial L}{\partial r}=\delta\cdot\frac{\partial L}{\partial f}$. Thus, we have $\frac{\partial L}{\partial r}<0$ iff $\frac{\partial L}{\partial f}<0$, which implies $r^T=f^T=0$. Since this scenario is not interesting, we focus on the case of $\frac{\partial L}{\partial r}=\frac{\partial L}{\partial f}=0$ as discussed above, instead of $(\frac{\partial L}{\partial r}<0,\frac{\partial L}{\partial f}=0)$ and $(\frac{\partial L}{\partial r}=0,\frac{\partial L}{\partial f}<0)$.

$\ddot{c}_2 > c_2' = \dfrac{(3-2b)a + (7+b-4b^2)c_1}{(10-b-4b^2)}$; and $B<0$ if $b<0.6937$ and c_2 satisfies

$\ddot{c}_2 < c_2 < \dfrac{(24-40b+10b^2+13b^3-6b^4)a + (8+32b-30b^2-13b^3+11b^4)c_1}{(32-8b-20b^2+5b^4)} \equiv c_2^w$. In contrast, $B>0$ if $b<0.6937$ and c_2 satisfies $c_2^w < c_2 < \tilde{c}_2$, or if $b>0.6937$ and $\ddot{c}_2 < c_2 < \tilde{c}_2$. These then imply $r^T < r^L$ if $W<0$ because $\dfrac{\partial L}{\partial r} = 0$ at $r = r^T$, $\dfrac{\partial L}{\partial r} = \frac{DW[R-\underline{R}]}{2} < 0$ at $(r = r^L, \delta = \breve{\breve{\delta}})$ by $W<0$, and $\dfrac{\partial^2 L}{\partial r^2} < 0$ at $(r = r^L, \delta = \breve{\breve{\delta}})$. Note that $A<0$ always holds, and hence $W<0$ holds if $b<0.6937$ and $\ddot{c}_2 < c_2 < c_2^w$, or if $c_2^w < c_2 < \tilde{c}_2$ and $D_1 < 0$, where

$$D_1 = \left\{ \begin{array}{l} \dfrac{\left[-(24b-40b^2+4b^3+17b^4-6b^5)a + (32+24b-48b^2+4b^3+15b^4-7b^5)c_1 - (32-8b^2-2b^4-b^5)c_2\right]\beta_1\left[\pi_2^T - \underline{U}_2\right]}{4(3-2b)(2-b^2)(4-3b^2)} \\ + \dfrac{(2-b)\left[-(24-40b+10b^2+13b^3-6b^4)a - (8+32b-30b^2-13b^3+11b^4)c_1 + (32-8b-20b^2+5b^4)c_2\right]\beta_2\left[\pi_1^T - \underline{U}_1\right]}{4(3-2b)(2-b^2)(4-3b^2)} \end{array} \right\},$$

and D_1 equals W evaluated at $(r = r^L, \delta = \breve{\breve{\delta}})$. Note that the former condition implies $B<0$, while the latter condition implies $B>0$ but B is small. Conversely, we have $r^T > r^L$ if $D_1 > 0$ for $\ddot{c}_2 < c_2 < \tilde{c}_2$.

Next, we examine the conditions for determining the sign of $\dfrac{\partial L}{\partial \delta} = \dfrac{DS\left[R-\underline{R}\right]}{4}$. Let $S = \bar{A}\beta_1\left[\pi_2^T - \underline{U}_2\right] + \bar{B}\beta_2\left[\pi_1^T - \underline{U}_1\right]$ where

$$\bar{A} = \frac{\left[(48-104b+36b^2+50b^3-22b^4-13b^5+6b^6)a - (16-104b+52b^2+66b^3-44b^4-13b^5+11b^6)c_1 - (32-16b^2-16b^3+22b^4-5b^6)c_2\right]}{2(3-2b)(2-b^2)(4-3b^2)}$$

and $\bar{B} = \dfrac{\left[-(24-76b+70b^2-11b^3-6b^4)a + (40-92b+6b^2+25b^3+b^4)c_1 - (16-16b-64b^2+36b^3+7b^4)c_2\right]}{2(3-2b)(4-3b^2)}$.

We have $\bar{A} < 0$ if $0.3588 < b < 0.5951$ and $c_2^s < c_2 < \tilde{c}_2$, or if $b > 0.5951$ for $\ddot{c}_2 < c_2 < \tilde{c}_2$; and $\bar{A} > 0$ if $b < 0.3588$ or if $0.3588 < b < 0.5951$ and $\ddot{c}_2 < c_2 < c_2^s$ with

$c_2^s = \dfrac{(48-104b+36b^2+50b^3-22b^4-13b^5+6b^6)a - (16-104b+52b^2+66b^3-44b^4-13b^5+11b^6)c_1}{(32-16b^2-16b^3+22b^4-5b^6)}$. On the other hand, we have $\bar{B} < 0$ if $b < 0.5037$; and $\bar{B} > 0$ if $b > 0.5037$ for $\ddot{c}_2 < c_2 < \tilde{c}_2$. Accordingly, we have $S > 0$, which implies $\dfrac{\partial L}{\partial \delta} > 0$ and $\delta^T > \breve{\breve{\delta}}$ due to $\dfrac{\partial L}{\partial \delta} = 0$ at $\delta = \delta^T$ and $\dfrac{\partial^2 L}{\partial \delta^2} < 0$ at $(r = r^L, \delta = \breve{\breve{\delta}})$, if $\bar{A} > 0$ and $\bar{B} > 0$, or if $D_2 > 0$, where

$$D_2 = \left\{ \begin{array}{l} \dfrac{\left[(48-104b+36b^2+50b^3-22b^4-13b^5+6b^6)a - (16-104b+52b^2+66b^3-44b^4-13b^5+11b^6)c_1 - (32-16b^2-16b^3+22b^4-5b^6)c_2\right]\beta_1\left[\pi_2^T - \underline{U}_2\right]}{2(3-2b)(2-b^2)(4-3b^2)} \\ + \dfrac{\left[-(24-76b+70b^2-11b^3-6b^4)a + (40-92b+6b^2+25b^3+b^4)c_1 - (16-16b-64b^2+36b^3+7b^4)c_2\right]\beta_2\left[\pi_1^T - \underline{U}_1\right]}{2(3-2b)(4-3b^2)} \end{array} \right\},$$

and D_2 equals S evaluated at $(r = r^L, \delta = \breve{\breve{\delta}})$. Moreover, $\bar{A} > 0$ and $\bar{B} > 0$ are implied by $0.5037 < b < 0.5951$ and $\ddot{c}_2 < c_2 < c_2^s$. In contrast, we will have $S < 0$, which implies $\frac{\partial L}{\partial \delta} < 0$ and $\delta^T < \breve{\breve{\delta}}$ if $\bar{A} < 0$ and $\bar{B} < 0$, or if $D_2 < 0$. Furthermore, $\bar{A} < 0$ and $\bar{B} < 0$ are implied by $0.3588 < b < 0.5037$ and $c_2^s < c_2 < \tilde{c}_2$.

All the conditions above imply $\frac{\partial L}{\partial r} < 0$ and $\frac{\partial L}{\partial \delta} > 0$ at $(r = r^L, \delta = \breve{\breve{\delta}})$ if $0.5037 < b < 0.5951$ and $\ddot{c}_2 < c_2 < \min\left\{c_2^w, c_2^s\right\}$, or if $D_1 < 0$ and $D_2 > 0$, while $\frac{\partial L}{\partial r} > 0$ and $\frac{\partial L}{\partial \delta} < 0$ at $(r = r^L, \delta = \breve{\breve{\delta}})$ if $0.3588 < b < 0.5037$, $c_2^s < c_2 < \tilde{c}_2$ and $D_1 > 0$. Moreover, we have

$$\delta^L - \breve{\breve{\delta}} = \frac{2\left(3b - 5b^2 + 2b^3\right)a + 2\left(4 - 5b - 2b^2 + b^3 + b^4\right)c_1 - 2\left(4 - 2b - 7b^2 + 3b^3 + b^4\right)c_2}{\left(3 - 2b\right)\left(2 - b^2\right)\left(4 - 3b^2\right)} > 0 \text{ if}$$

$0.4343 < b < 0.6309$ and $\ddot{c}_2 < c_2 < c_2^d$, or if $b > 0.6309$ for $\ddot{c}_2 < c_2 < \tilde{c}_2$, where $c_2^d = \frac{\left(3b - 5b^2 + 2b^3\right)a + \left(4 - 5b - 2b^2 + b^3 + b^4\right)c_1}{\left(4 - 2b - 7b^2 + 3b^3 + b^4\right)} < c_2^w$. Accordingly, some calculations yield $\prod^B > \prod^L$ if $0.5037 < b < 0.5951$ and $\ddot{c}_2 < c_2 < \min\left\{c_2^d, c_2^s\right\}$, or if $D_1 < 0$ and $D_2 > 0$; and $\prod^B < \prod^L$ if $D_1 > 0$ and $D_2 < 0$. The former proves Proposition 4(iiic) and the latter proves Proposition 4(iiib). Moreover, we have $\pi_2^T > 0$ as assumed in our model, and $\pi_2^L = 0$ as proved by Proposition B. Thus, $\pi_2^B \equiv \pi_2^T > \pi_2^L$. However, the relative sizes of $\pi_1^B \equiv \pi_1^T$ and π_1^L are uncertain. This is because we cannot express δ^L and δ^T as single-variable functions of unit-fee rate r as Proposition 4(iiia) does. Instead, they are multi-variable functions of r and f. Therefore, we cannot determine the relative sizes of π_1^B and π_1^L.

Finally, it is worth mentioning that our analyses above can be generalized by finding $\breve{\breve{\delta}}$ for any r with $\breve{\breve{\delta}}(r) \in \arg\left\{\prod(r, \breve{\breve{\delta}}) - \prod(r^L, \delta^L) = 0\right\}$, instead of finding $\breve{\breve{\delta}}$ given r^L. Afterwards, we can compare the relationships among $\breve{\breve{\delta}}(r)$, δ^T and δ^L to determine the relationship between $\prod\left(r^T, \delta^T\right)$ and $\prod\left(r^L, \delta^L\right)$. Specifically,

$$\breve{\breve{\delta}}(r) \in \arg\left\{\begin{array}{l} \frac{1}{2}\left(a - c_1 - r - b\breve{\breve{\delta}}\right)^2 + \frac{\breve{\breve{\delta}}}{2}[(2-b)(a-r) - (2-b^2)\breve{\breve{\delta}} + bc_1 - 2c_2] + \frac{r}{2}\left[a - c_1 - r + (2-b)\breve{\breve{\delta}}\right] \\ -\left\{\frac{1}{2}\left(a - c_1 - r^L - b\delta^L\right)^2 + \frac{\delta^L}{2}[(2-b)(a-r^u) - (2-b^2)\delta^L + bc_1 - 2c_2] + \frac{r^L}{2}\left[a - c_1 - r^L + (2-b)\delta^L\right]\right\} \end{array}\right\} = 0.$$

Actually, we can obtain that $\delta^L > (<)\ \breve{\delta}$ with $\breve{\delta}(r) = \frac{2\left(1-b\right)a + br + 2bc_1 - 2c_2}{\left(4 - 3b^2\right)}$ iff

$$r < (>) \frac{\left(12b - 14b^2 + b^3 + 2b^4\right)a + 2\left(8 - 12b - 2b^2 + 3b^3 + b^4\right)c_1 - \left(8 - 16b^2 + 4b^3 + 3b^4\right)c_2}{2b\left(3 - 2b\right)\left(2 - b^2\right)}. \text{ Thus, if}$$

$$r < \frac{\left(12b-14b^2+b^3+2b^4\right)a+2\left(8-12b-2b^2+3b^3+b^4\right)c_1-\left(8-16b^2+4b^3+3b^4\right)c_2}{2b\left(3-2b\right)\left(2-b^2\right)}$$, then $\delta^L > \breve{\delta}$,

which implies $\prod^B < \prod^L$ if $\delta^T < \breve{\breve{\delta}}$, and $\prod^B > \prod^L$ if $\breve{\breve{\delta}} < \delta^T < \delta^L$. In contrast, for

$$r > \frac{\left(12b-14b^2+b^3+2b^4\right)a+2\left(8-12b-2b^2+3b^3+b^4\right)c_1-\left(8-16b^2+4b^3+3b^4\right)c_2}{2b\left(3-2b\right)\left(2-b^2\right)}$$, we have $\delta^L < \breve{\delta}$,

which implies $\prod^B < \prod^L$ if $\delta^T < \delta^L$, and $\prod^B > \prod^L$ if $\delta^L < \delta^T < \breve{\breve{\delta}}$.

Let S_1 be the value of $\frac{\partial L}{\partial \delta}$ at $(r = r^{s1},\ \delta = \breve{\breve{\delta}})$, and S_2 be the value of $\frac{\partial L}{\partial \delta}$ at $(r = r^{s2},\ \delta = \delta^L)$, where r^{s1} and r^{s2} solve the following equations:

$$r^{s1} \in \arg\left\{\beta_1\left[-(a-c_1)+(2+b)\breve{\breve{\delta}}\right]\left[\pi_2^T-\underline{U}_2\right]+\beta_2\left[(a-c_1)-2r-(2-b)\breve{\breve{\delta}}\right]\left[\pi_1^T-\underline{U}_1\right]=0\right\},$$

$$r^{s2} \in \arg\left\{\beta_1\left[-(a-c_1)+(2+b)\delta^L\right]\left[\pi_2^T-\underline{U}_2\right]+\beta_2\left[(a-c_1)-2r-(2-b)\delta^L\right]\left[\pi_1^T-\underline{U}_1\right]=0\right\},$$

$$S_1 = \beta_1\left[-2b(a-c_1)+(2+b)r^{s1}+2b^2\breve{\breve{\delta}}\right]\left[\pi_2^T-\underline{U}_2\right]+\beta_2\left[2(2-b)\mathrm{a}-(2-b)r^{s1}-4\left(2-b^2\right)\breve{\breve{\delta}}+2bc_1-4c_2\right]\left[\pi_1^T-\underline{U}_1\right],$$ and

$$S_2 = \beta_1\left[-2b(a-c_1)+(2+b)r^{s2}+2b^2\delta^L\right]\left[\pi_2^T-\underline{U}_2\right]+\beta_2\left[2(2-b)\mathrm{a}-(2-b)r^{s2}-4\left(2-b^2\right)\delta^L+2bc_1-4c_2\right]\left[\pi_1^T-\underline{U}_1\right].$$

In summary, for $r^{s2} < \frac{\left(12b-14b^2+b^3+2b^4\right)a+2\left(8-12b-2b^2+3b^3+b^4\right)c_1-\left(8-16b^2+4b^3+3b^4\right)c_2}{2b\left(3-2b\right)\left(2-b^2\right)}$,

we have $\prod^B < \prod^L$ if $S_1 < 0$, and $\prod^B > \prod^L$ if $S_1 > 0$ and $S_2 < 0$. When

$$r^{s2} > \frac{\left(12b-14b^2+b^3+2b^4\right)a+2\left(8-12b-2b^2+3b^3+b^4\right)c_1-\left(8-16b^2+4b^3+3b^4\right)c_2}{2b\left(3-2b\right)\left(2-b^2\right)}$$, we have $\prod^B < \prod^L$ if

$S_2 < 0$, and $\prod^B > \prod^L$ if $S_1 < 0$ and $S_2 > 0$. These imply that Proposition 4(iiib)-(iiic) remain true under the conditions stated above.

Case b: Suppose $q_1^L = \delta$ and $q_2^L = \delta$ (or $q_1^T = \delta$ and $q_2^T = \delta$) as given in Lemma 1(iii). Under these circumstances, the total payoff of the port authority and the two operators in our model is $\prod = \pi_1 + \pi_2 + R = \delta\left[2a - 2(1+b)\delta - c_1 - c_2\right]$ by Lemma 1(iii), and is $\prod^L = \frac{(a-c_1)(a-c_2)}{2(1+b)}$ in Liu et al. (2018) for $c_2 \in [c_1, \ddot{c}_2]$ by Proposition B(i).

By some calculations, we have

$$\frac{\partial \prod}{\partial \delta} = 2a - 4(1+b)\delta - c_1 - c_2 > (<)\ 0 \text{ iff } \delta < (>)\frac{2a-(c_1+c_2)}{4(1+b)} \equiv \hat{\delta}, \text{ and } \frac{\partial^2 \prod}{\partial \delta^2} = -4(1+b) < 0$$

for all $\delta \geq 0$. Moreover, $\hat{\delta} - \delta^L = \frac{2a-(c_1+c_2)}{4(1+b)} - \frac{(a-c_2)}{2(1+b)} = \frac{(c_2-c_1)}{4(1+b)} > 0$. These imply that $\prod^B < \prod^L$ if $\delta^T < \delta^L < \hat{\delta}$, $\prod^B > \prod^L$ if $\delta^L < \delta^T < \hat{\delta}$, and the relative sizes of $\prod^B$ and $\prod^L$ are uncertain if $\delta^L < \hat{\delta} < \delta^T$. Thus, we will analyze the relationships among δ^L, δ^T and $\hat{\delta}$ below using the first-order conditions of $(r^T,\ f^T,\ \delta^T)$.

As in Case a, we start with the interior solution derived by $\frac{\partial L}{\partial f}=0$, $\frac{\partial L}{\partial r}=0$ and

$$\frac{\partial L}{\partial \delta}=D\cdot\left\{(1-\beta_1-\beta_2)\frac{\partial R}{\partial \delta}\left[\pi_1^T-\underline{U}_1\right]\left[\pi_2^T-\underline{U}_2\right]+\beta_1\frac{\partial \pi_1^T}{\partial \delta}\left[R-\underline{R}\right]\left[\pi_2^T-\underline{U}_2\right]+\beta_2\frac{\partial \pi_2^T}{\partial \delta}\left[R-\underline{R}\right]\left[\pi_1^T-\underline{U}_1\right]\right\}.$$

Thus, by $\hat{\delta}>\delta^L$, we will have $\delta^L<\hat{\delta}<\delta^T$ if $\frac{\partial L}{\partial \delta}>0$ at $\delta=\hat{\delta}$; and $\delta^T<\delta^L<\hat{\delta}$ or $\delta^L<\delta^T<\hat{\delta}$ if $\frac{\partial L}{\partial \delta}<0$ at $\delta=\hat{\delta}$. Therefore, it remains to identify the sign of $\frac{\partial L}{\partial \delta}$ as follows.

Note that $\frac{\partial L}{\partial f}=0$ implies

$2(1-\beta_1-\beta_2)\left[\pi_1^T-\underline{U}_1\right]\left[\pi_2^T-\underline{U}_2\right]=\beta_1\left[R-\underline{R}\right]\left[\pi_2^T-\underline{U}_2\right]+\beta_2\left[R-\underline{R}\right]\left[\pi_1^T-\underline{U}_1\right]$. Substituting it into $\frac{\partial L}{\partial \delta}$ yields $\frac{\partial L}{\partial \delta}=D\cdot\frac{\left[R-\underline{R}\right]}{2}\cdot M$, where

$M=\beta_1\left[2a-4(1+b)\delta-2c_1\right]\left[\pi_2^T-\underline{U}_2\right]+\beta_2\left[2a-4(1+b)\delta-2c_2\right]\left[\pi_1^T-\underline{U}_1\right]$ with

$$\frac{\partial M}{\partial \delta}=\left\{\begin{array}{l}-4(1+b)\left\{\beta_1\left[\pi_2^T-\underline{U}_2\right]+\beta_2\left[\pi_1^T-\underline{U}_1\right]\right\}+2(\beta_1+\beta_2)\left[a-2(1+b)\delta-c_1\right]\left[a-2(1+b)\delta-c_2\right]\\-2r\left\{\beta_1\left[a-2(1+b)\delta-c_1\right]+\beta_2\left[a-2(1+b)\delta-c_2\right]\right\}\end{array}\right\}.$$

We have $M=0$ and $\frac{\partial M}{\partial \delta}<0$ at r^T and δ^T. Moreover, at $\delta=\hat{\delta}$,

$$M=\beta_1\left[2a-4(1+b)\delta-2c_1\right]\left[\pi_2^T-\underline{U}_2\right]+\beta_2\left[2a-4(1+b)\delta-2c_2\right]\left[\pi_1^T-\underline{U}_1\right]$$
$$=(c_2-c_1)\left\{\beta_1\left[\frac{(2a+c_1-3c_2-4r^T)(2a-c_1-c_2)}{16(1+b)}-f^T-\underline{U}_2\right]-\beta_2\left[\frac{(2a-3c_1+c_2-4r^T)(2a-c_1-c_2)}{16(1+b)}-f^T-\underline{U}_1\right]\right\}.$$

Thus, if $M<0$ at $\delta=\hat{\delta}$, then $\frac{\partial L}{\partial \delta}<0$ at $\delta=\hat{\delta}$, and hence $\delta^T<\hat{\delta}$. On the other hand, at $\delta=\delta^L$,

$$M=2\beta_1(c_2-c_1)\left[\frac{(a-c_2-2r^T)(a-c_2)}{4(1+b)}-f^T-\underline{U}_2\right]>0 \text{ and}$$

$\frac{\partial M}{\partial \delta}=\left\{-4(1+b)\left\{\beta_1\left[\pi_2^T-\underline{U}_2\right]+\beta_2\left[\pi_1^T-\underline{U}_1\right]\right\}-2r\beta_1(c_2-c_1)\right\}<0$. We then have $\delta^L<\delta^T$. Thus, $\delta^L<\delta^T<\hat{\delta}$ and hence $\prod^T>\prod^L$. In contrast, if $M>0$ at $\delta=\hat{\delta}$, then

$$\frac{\partial M}{\partial \delta}=-4(1+b)\left\{\beta_1\left[\pi_2^T-\underline{U}_2\right]+\beta_2\left[\pi_1^T-\underline{U}_1\right]\right\}-\frac{1}{2}(\beta_1+\beta_2)(c_2-c_1)^2-r\left[\beta_1(c_2-c_1)-\frac{1}{2}\beta_2(c_2-c_1)\right]<0,$$

and hence $\delta^T>\hat{\delta}$. Moreover, we have $\prod(\delta=\frac{a}{1+b})-\prod(\delta=\delta^L)=\frac{-2a(c_1+c_2)-(a-c_1)(a-c_2)}{2(1+b)}<0$.

This implies the existence of $\hat{\hat{\delta}} = \frac{(a-c_1)}{2(1+b)}$ with $\hat{\delta} < \hat{\hat{\delta}} < \frac{a}{1+b}$ to ensure $\prod(\delta = \hat{\hat{\delta}}) = \prod(\delta = \delta^L)$. On the other hand, we have $\delta^L < \hat{\delta}$ as shown in the beginning, and $\delta^T < \hat{\hat{\delta}}$ implied by $M = -2\beta_2(c_2 - c_1)\left[\frac{(a-c_1-2r^T)(a-c_1)}{4(1+b)} - f^T - \underline{U}_1\right] < 0$ at $\delta = \hat{\hat{\delta}}$. Thus, we obtain $\delta^L < \hat{\delta} < \delta^T < \hat{\hat{\delta}} < \frac{a}{1+b}$, and hence $\prod^B > \prod^L$.

It remains to show that $\pi_1^B > \pi_1^L > 0$ and $\pi_2^B > \pi_2^L = 0$. Since $\pi_2^B > 0$ as assumed in our model and $\pi_2^L = 0$ as shown by Proposition B, it remains to prove $\pi_1^B > \pi_1^L$. Under $c_2 \in (c_1, \ddot{c}_2]$, the optimal contract is the unit-fee scheme in Liu et al. (2018), and can be either the two-part tariff or the unit-fee scheme in our model.[20] In the following, we will only compare the optimal unit-fee contract and minimum throughput (r^T, δ^T) in our model with the optimal contract (r^L, δ^L) in Liu et al. (2018) as the same arguments can be applied to compare our other types of contracts with (r^L, δ^L) in Liu et al. (2018). Moreover, we will compare (r^T, δ^T) and (r^L, δ^L) through the level curve of π_1 as follows. Given any unit-fee scheme (r) and minimum throughput requirement (δ), operator 1's profit is $\pi_1 = [a - (1+b)\delta - c_1 - r]\delta$ by Lemma 1(iii) with $\frac{\partial \pi_1}{\partial \delta} = a - 2(1+b)\delta - c_1 - r < 0$ due to $a - 2(1+b)\delta - c_1 - r < a - 2(1+b)\delta_2 - c_1 - r = \frac{-b(a-c_1-r)}{2+b} < 0$ and $\frac{\partial \pi_1}{\partial r} = -\delta < 0$. Thus, given the level curve of π_1, $\pi_1 = k$ for some fixed real number k, the implicit function theorem implies that we can express δ as a function of r with $\frac{\partial \delta}{\partial r} = -\frac{\partial \pi_1}{\partial r} \Big/ \frac{\partial \pi_1}{\partial \delta} = \frac{\delta}{a - 2(1+b)\delta - c_1 - r} < 0$. This means that when r increases by one unit, δ needs to decrease by $\left|\frac{\delta}{a - 2(1+b)\delta - c_1 - r}\right|$ units to ensure $\pi_1 = k$ remains unchanged. Moreover, at (r^L, δ^L), we have

$$\left.\frac{\partial \delta}{\partial r}\right|_{\{r^L, \delta^L\}} = \frac{\delta^L}{a - 2(1+b)\delta^L - c_1 - r^L} = \frac{-(a-c_2)}{(1+b)(a+2c_1-3c_2)} < 0$$

by $a + 2c_1 - 3c_2 > a + 2c_1 - 3c_2'' > a + 2c_1 - 3\ddot{c}_2 > 0$ with $c_2'' = \frac{a + 2(1+b)c_1}{3+2b} > \ddot{c}_2$. On the other hand, as shown by Proposition B(i), the optimal minimum throughput requirement δ^L can also be expressed

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[20]In the proof of Proposition 2, we demonstrate $\frac{\partial L}{\partial r} = \delta \cdot \frac{\partial L}{\partial f}$, $\frac{\partial L}{\partial r} = \frac{\partial L}{\partial f} = 0$ and $\frac{\partial L}{\partial \delta} = 0$, allowing us to solve the three endogenous variables, r, δ, and f using two equations. Thus, the optimal contract can be either a two-part tariff scheme or a unit-fee scheme.

as a function of the unit-fee rate r with $\delta^L(r)=\frac{a-c_2-r}{1+b}$ and $\frac{\partial\delta^L(r)}{\partial r}=\frac{-1}{1+b}<0$. Some calculations yield

$$\frac{\partial\delta^L(r)}{\partial r}-\left.\frac{\partial\delta}{\partial r}\right|_{\{r^L,\delta^L\}}=\frac{-1}{1+b}-\frac{-(a-c_2)}{(1+b)(a+2c_1-3c_2)}=\frac{2(c_2-c_1)}{(1+b)(a+2c_1-3c_2)}>0.$$

This implies that starting from (r^L,δ^L), given that r decreases by one unit, the value of δ needs to increase to ensure $\pi_1=k$ will be greater than the value that $\delta^L(r)$ increases to keep $\pi_1=k$. Moreover, as shown above, we have $\delta^L<\delta^T$ and $\delta^T<\frac{a-c_2-r}{1+b}$ by $[a-(1+b)\delta-c_2-r]\delta>0$, which then implies $r^T<r^L$ and $\delta^T(r)<\frac{a-c_2-r}{1+b}=\delta^L(r)$. By letting $\varepsilon=\frac{(a-c_2)}{(1+b)(a+2c_1-3c_2)}$, we can have

$$\pi_1^L\equiv\pi_1(r^L,\delta^L)=\pi_1(r^T,\delta^L+\varepsilon(r^L-r^T))<\pi_1(r^T,\delta^L+\frac{r^L-r^T}{1+b})=\pi_1(r^T,\delta^L(r^T))<\pi_1(r^T,\delta^T)\equiv\pi_1^B.$$

Here $\pi_1(r^L,\delta^L)=\pi_1(r^T,\delta^L+\varepsilon(r^L-r^T))$ holds because the changes in r and δ can ensure π_1^L remains unchanged if $\frac{\partial\delta}{\partial r}=\frac{-(a-c_2)}{(1+b)(a+2c_1-3c_2)}=-\varepsilon$ is satisfied,[21] and $\pi_1(r^T,\delta^L+\varepsilon(r^L-r^T))<\pi_1(r^T,\delta^L+\frac{r^L-r^T}{1+b})$ holds due to $\frac{\partial\pi_1}{\partial\delta}<0$ and $\varepsilon-\frac{1}{1+b}=\frac{2(c_2-c_1)}{(1+b)(a+2c_1-3c_2)}>0$. These then imply $\pi_1^B>\pi_1^L>0$. All the above prove Proposition 4(iiia).

<u>Proof of Proposition 6</u>: According to Lemma 1, there are three cases as follows. Once more, we concentrate on the situations of $\pi_1^T>\underline{U}_1$, $\pi_2^T>\underline{U}_2$ and $W^S>\underline{W}$ to avoid uninteresting solutions.

<u>Case 1</u>: Suppose $\delta\in[0,\delta_1]$ with $\delta_1\equiv\frac{a-r}{2+b}+\frac{bc_1-2c_2}{4-b^2}$. We have $\pi_1^T=(q_1^T)^2-f$, $\pi_2^T=(q_2^T)^2-f$, and $W^S=(a-c_1-r)q_1+(a-c_2-r)q_2-\frac{1}{2}[q_1^2+2bq_1q_2+q_2^2]-2f$. Accordingly, the problem in (17) becomes

$$\max_{r\geq0,f\geq0,\delta\geq0}\,[W^S-\underline{W}]^{(1-\beta_1-\beta_2)}[\pi_1^T-\underline{U}_1]^{\beta_1}[\pi_2^T-\underline{U}_2]^{\beta_2}$$

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[21]This equality holds when r^L and r^T are close to each other. If they are not, then we will have $\pi_1(r^L,\delta^L)<\pi_1(r^T,\delta^L+\varepsilon(r^L-r^T))$ by the convexity of the level curves of π_1, and our conclusions still hold.

$$\text{s.t. } 0 \le \delta \le \delta_1,\ 0 \le r < \bar{r},\ f \ge 0,\ \pi_1^T \ge \underline{U}_1,\ \pi_2^T \ge \underline{U}_2 \text{ and } W^S \ge \underline{W}. \qquad \text{(A47)}$$

The same arguments as in Case 1 of Proposition 2 can be applied here. Define $L = \left[W^{\mathrm{S}} - \underline{W}\right]^{(1-\beta_1-\beta_2)} \left[\pi_1^T - \underline{U}_1\right]^{\beta_1} \left[\pi_2^T - \underline{U}_2\right]^{\beta_2}$, and the corresponding Kuhn-Tucker conditions associated with problem (A47) are

$$\frac{\partial L}{\partial r} = D \cdot \left\{ (1-\beta_1-\beta_2)\frac{\partial W^{\mathrm{S}}}{\partial r}\left[\pi_1^T - \underline{U}_1\right]\left[\pi_2^T - \underline{U}_2\right] - \frac{2\beta_1 q_1^T}{2+b}\left[W^{\mathrm{S}} - \underline{W}\right]\left[\pi_2^T - \underline{U}_2\right] - \frac{2\beta_2 q_2^T}{2+b}\left[W^{\mathrm{S}} - \underline{W}\right]\left[\pi_1^T - \underline{U}_1\right] \right\},\ r \cdot \frac{\partial L}{\partial r} = 0,$$

$$\frac{\partial L}{\partial f} = D \cdot \left\{ -2(1-\beta_1-\beta_2)\left[\pi_1^T - \underline{U}_1\right]\left[\pi_2^T - \underline{U}_2\right] - \beta_1\left[W^{\mathrm{S}} - \underline{W}\right]\left[\pi_2^T - \underline{U}_2\right] - \beta_2\left[W^{\mathrm{S}} - \underline{W}\right]\left[\pi_1^T - \underline{U}_1\right] \right\} \le 0,\ f \cdot \frac{\partial L}{\partial f} = 0,$$

where $D = \left\{ [W^{\mathrm{S}} - \underline{W}]^{-(\beta_1+\beta_2)} [\pi_1^T - \underline{U}_1]^{-(1-\beta_1)} [\pi_2^T - \underline{U}_2]^{-(1-\beta_2)} \right\} > 0$. Since $\frac{\partial W^{\mathrm{S}}}{\partial r} = \frac{-(3+b)\left(q_1^T + q_2^T\right)}{2+b} < 0$ and $\frac{\partial W^{\mathrm{S}}}{\partial f} = -2 < 0$, we have $\frac{\partial L}{\partial r} < 0$ and $\frac{\partial L}{\partial f} < 0$. This implies $r^s = 0$ and $f^s = 0$, and hence there exists no meaningful fee-charging contract. Instead, it is optimal for the port authority to subsidize the operators.

<u>Case 2</u>: Suppose $\delta \in (\delta_1, \delta_2]$ with $\delta_2 \equiv \frac{a-c_1-r}{2+b} > \delta_1$. We have $\pi_1^T = \left(q_1^T\right)^2 - f$ with $q_1^T \equiv \frac{a-c_1-r-b\delta}{2} > \delta_2$, $\pi_2^T = \frac{\delta}{2}[(2-b)(a-r) - (2-b^2)\delta + bc_1 - 2c_2] - f$ and $W^{\mathrm{S}} = (a-c_1-r)q_1 + (a-c_2-r)\delta - \frac{1}{2}[q_1^2 + 2bq_1\delta + \delta^2] - 2f$. Accordingly, the problem in (17) becomes

$$\max_{r\ge 0, f\ge 0, \delta\ge 0}\ [W^{\mathrm{S}} - \underline{W}]^{(1-\beta_1-\beta_2)} [\pi_1^T - \underline{U}_1]^{\beta_1} [\pi_2^T - \underline{U}_2]^{\beta_2}$$

$$\text{s.t. } \delta_1 < \delta \le \delta_2,\ 0 \le r < \bar{r},\ f \ge 0,\ \pi_1^T \ge \underline{U}_1,\ \pi_2^T \ge \underline{U}_2 \text{ and } W^S \ge \underline{W}. \qquad \text{(A48)}$$

The corresponding Kuhn-Tucker conditions in problem (48) are as follows.

$$\frac{\partial L}{\partial r} = D \cdot \left\{ (1-\beta_1-\beta_2)\frac{\partial W^{\mathrm{S}}}{\partial r}\left[\pi_1^T - \underline{U}_1\right]\left[\pi_2^T - \underline{U}_2\right] - \beta_1 q_1^T\left[\pi_2^T - \underline{U}_2\right]\left[W^{\mathrm{S}} - \underline{W}\right] - \frac{2-b}{2}\beta_2\delta\left[\pi_1^T - \underline{U}_1\right]\left[W^{\mathrm{S}} - \underline{W}\right] \right\} \le 0,\ r \cdot \frac{\partial L}{\partial r} = 0,$$

$$\frac{\partial L}{\partial f} = D \cdot \left\{ -2(1-\beta_1-\beta_2)\left[\pi_1^T - \underline{U}_1\right]\left[\pi_2^T - \underline{U}_2\right] - \beta_1\left[W^{\mathrm{S}} - \underline{W}\right]\left[\pi_2^T - \underline{U}_2\right] - \beta_2\left[W^{\mathrm{S}} - \underline{W}\right]\left[\pi_1^T - \underline{U}_1\right] \right\} \le 0,\ f \cdot \frac{\partial L}{\partial f} = 0,$$

$$\frac{\partial L}{\partial \delta} = D \cdot \left\{ \begin{aligned} &(1-\beta_1-\beta_2)\frac{\partial W^{\mathrm{S}}}{\partial \delta}\left[\pi_1^T - \underline{U}_1\right]\left[\pi_2^T - \underline{U}_2\right] - b\beta_1 q_1^T\left[\pi_2^T - \underline{U}_2\right]\left[W^{\mathrm{S}} - \underline{W}\right] \\ &+ \frac{1}{2}\beta_2\left[(2-b)(a-r) - 2(2-b^2)\delta + bc_1 - 2c_2\right]\left[\pi_1^T - \underline{U}_1\right]\left[W^{\mathrm{S}} - \underline{W}\right] \le 0 \end{aligned} \right\} \le 0,\ \delta \cdot \frac{\partial L}{\partial \delta} = 0,$$

where $D = \left\{ [W^{\mathrm{S}} - \underline{W}]^{-(\beta_1+\beta_2)} [\pi_1^T - \underline{U}_1]^{-(1-\beta_1)} [\pi_2^T - \underline{U}_2]^{-(1-\beta_2)} \right\} > 0$. Since

$\frac{\partial W^S}{\partial r} = -\frac{1}{2}\left(a - c_1 - r\right) - \frac{1}{2}q_1^T - \frac{2-b}{2}\delta < 0$ and $\frac{\partial W^S}{\partial f} = -2 < 0$, we have $\frac{\partial L}{\partial r} < 0$ and $\frac{\partial L}{\partial f} < 0$ by $\pi_i^T > \underline{U}_i$ and $W^S > \underline{W}$ for $i = 1, 2$. These imply $r^s = 0$ and $f^s = 0$. Again, the subsidy is the best choice for all three parties. However, we need to solve δ^s by $\delta \cdot \frac{\partial L}{\partial \delta} = 0$, which is consistent with $r^s = 0$ and $f^s = 0$.

The sign of $\frac{\partial L}{\partial \delta}$ depends on the values of the three parts: $(1-\beta_1-\beta_2)\frac{\partial W^S}{\partial \delta}\left[\pi_1^T - \underline{U}_1\right]\left[\pi_2^T - \underline{U}_2\right]$, $b\beta_1 q_1^T\left[\pi_2^T - \underline{U}_2\right]\left[W^S - \underline{W}\right]$ and $\frac{1}{2}\beta_2\left[(2-b)(a-r) - 2(2-b^2)\delta + bc_1 - 2c_2\right]\left[\pi_1^T - \underline{U}_1\right]\left[W^S - \underline{W}\right]$. Since we cannot assess the relative magnitudes of these three terms, we cannot determine the sign of $\frac{\partial L}{\partial \delta}$, nor can we determine the sign of $\frac{\partial^2 L}{\partial \delta^2}$ for $\delta \in (\delta_1, \delta_2]$.[22] However, if the optimal δ^S exists, then we must have $\delta^s \in (\delta_1, \delta_2]$ and $\frac{\partial L}{\partial \delta} = 0$ at $\delta = \delta^s$. In other words, if the model's parameters $(\beta_1, \beta_2, b, c_1, c_2, \underline{U}_1, \underline{U}_2, \underline{W})$ satisfy condition G with

$$G = \left\{ \begin{array}{l} \frac{(2-b)a + bc_1 - 2c_2}{4-b^2} < \delta^s < \frac{a-c_1}{2+b}, \text{ and} \\ \left\{ \begin{array}{l} (1-\beta_1-\beta_2)\left\{ \begin{array}{l} -\frac{1}{4}\left(4-3b^2\right)\left[\pi_1^T - \underline{U}_1\right]\left[\pi_2^T - \underline{U}_2\right] - \frac{1}{8}b\left(a - c_1 - b\delta^s\right)\left[\left(4-3b\right)a + 3bc_1 - 4c_2 - \left(4-3b^2\right)\delta^s\right]\left[\pi_2^T - \underline{U}_2\right] \\ +\frac{1}{8}\left[(2-b)a - 2(2-b^2)\delta^s + bc_1 - 2c_2\right]\left[\left(4-3b\right)a + 3bc_1 - 4c_2 - \left(4-3b^2\right)\delta^s\right]\left[\pi_1^T - \underline{U}_1\right] \end{array} \right\} \\ -b\beta_1\left\{ \begin{array}{l} \frac{1}{4}\left(a - c_1 - b\delta^s\right)\left[(2-b)a - 2(2-b^2)\delta^s + bc_1 - 2c_2\right]\left[W^S - \underline{W}\right] \\ +\frac{1}{8}\left(a - c_1 - b\delta^s\right)\left[\left(4-3b\right)a + 3bc_1 - 4c_2 - \left(4-3b^2\right)\delta^s\right]\left[\pi_2^T - \underline{U}_2\right] - \frac{b}{2}\left[\pi_2^T - \underline{U}_2\right]\left[W^S - \underline{W}\right] \end{array} \right\} \\ +\frac{1}{2}\beta_2\left\{ \begin{array}{l} -2(2-b^2)\left[\pi_1^T - \underline{U}_1\right]\left[W^S - \underline{W}\right] - \frac{1}{2}b\left(a - c_1 - b\delta^s\right)\left[(2-b)a - 2(2-b^2)\delta^s + bc_1 - 2c_2\right]\left[W^S - \underline{W}\right] \\ +\frac{1}{4}\left[(2-b)a - 2(2-b^2)\delta^s + bc_1 - 2c_2\right]\left[\pi_1^T - \underline{U}_1\right]\left[\left(4-3b\right)a + 3bc_1 - 4c_2 - \left(4-3b^2\right)\delta^s\right] \end{array} \right\} \end{array} \right\} < 0 \end{array} \right\}, \tag{A49}$$

then the optimal contract is $r^T = 0$, $f^T = 0$ and

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[22]For instance, we have $\frac{\partial W^s}{\partial \delta} = -\frac{3b\left(a - c_1 - r\right)}{4} + \left(a - c_2 - r\right) - \frac{\left(4 - 3b^2\right)\delta}{4} > 0$ at $\delta = \delta_1$ iff $c_2 < \frac{\left(2-b\right)^2\left(a - r\right) + 4bc_1}{4 + b^2}$. Even though $\frac{\partial W^s}{\partial \delta} > 0$, we cannot determine the sign of $\frac{\partial L}{\partial \delta}$. Similar arguments can be applied to the other terms.

$$\delta^{s1} \in \arg\left\{ \left\{ \begin{array}{c} (1-\beta_1-\beta_2)\dfrac{(4-3b)a+3bc_1-4c_2-(4-3b^2)\delta}{4}\left[\pi_1^T-\underline{U}_1\right]\left[\pi_2^T-\underline{U}_2\right]-b\beta_1 q_1^T\left[\pi_2^T-\underline{U}_2\right]\left[W^{\mathrm{S}}-\underline{W}\right] \\ +\dfrac{1}{2}\beta_2\left[(2-b)a-2(2-b^2)\delta+bc_1-2c_2\right]\left[\pi_1^T-\underline{U}_1\right]\left[W^{\mathrm{S}}-\underline{W}\right] \end{array} \right\} =0 \right\}.$$

(A50)

<u>Case 3</u>: Suppose $\delta \in (\delta_2, \frac{a}{1+b})$ with $\delta_2 \equiv \frac{a-c_1-r}{2+b} > \delta_1$. We have $\pi_1^T = \left[a-(1+b)\delta-c_1-r\right]\delta - f$, $\pi_2^T = \left[a-(1+b)\delta-c_2-r\right]\delta - f$ and $W^{\mathrm{S}} = \left[2(a-r)-c_1-c_2\right]\delta-(1+b)\delta^2-2f$. Accordingly, the problem in (17) becomes

$$\max_{r\geq 0, f\geq 0, \delta\geq 0}\ [W^{\mathrm{S}}-\underline{W}]^{(1-\beta_1-\beta_2)}[\pi_1^T-\underline{U}_1]^{\beta_1}[\pi_2^T-\underline{U}_2]^{\beta_2}$$

$$\text{s.t. } \delta_2<\delta<\frac{a}{1+b},\ 0\leq r<\overline{r},\ f\geq 0,\ \pi_1^T\geq \underline{U}_1,\ \pi_2^T\geq \underline{U}_2 \text{ and } W^{\mathrm{S}}\geq \underline{W}. \qquad \text{(A51)}$$

Its Lagrange function is

$$L=\left\{ \begin{array}{l} [W^{\mathrm{S}}-\underline{W}]^{(1-\beta_1-\beta_2)}\left[\pi_1^T-\underline{U}_1\right]^{\beta_1}\left[\pi_2^T-\underline{U}_2\right]^{\beta_2}+\lambda_1(\delta-\delta_2)+\lambda_2\left(\dfrac{a}{1+b}-\delta\right)+\lambda_3(\overline{r}-r) \\ +\lambda_4\left(\pi_1^T-\underline{U}_1\right)+\lambda_5\left(\pi_2^T-\underline{U}_2\right)+\lambda_6\left[W-\underline{W}\right] \end{array} \right\},$$

where λ_1, λ_2, λ_3, λ_4, λ_5 and λ_6 are the respective Lagrange multipliers associated with the five inequality constraints in (A51). Again, as in Case 2, we have $\lambda_3^*=\lambda_4^*=\lambda_5^*=\lambda_6^*=0$ and $\lambda_1^*=\lambda_2^*=0$ by $\delta_2<\delta<\frac{a}{1+b}$. Thus, the Kuhn-Tucker conditions for (r^s, f^s, δ^s) are given below.

$$\frac{\partial L}{\partial r}=D\cdot\left\{-2(1-\beta_1-\beta_2)\delta\left[\pi_1^T-\underline{U}_1\right]\left[\pi_2^T-\underline{U}_2\right]-\beta_1\delta[W^{\mathrm{S}}-\underline{W}]\left[\pi_2^T-\underline{U}_2\right]-\beta_2\delta[W^{\mathrm{S}}-\underline{W}]\left[\pi_1^T-\underline{U}_1\right]\right\}\leq 0,\ r\cdot\frac{\partial L}{\partial r}=0,$$

$$\frac{\partial L}{\partial f}=D\cdot\left\{-2(1-\beta_1-\beta_2)\left[\pi_1^T-\underline{U}_1\right]\left[\pi_2^T-\underline{U}_2\right]-\beta_1[W^{\mathrm{S}}-\underline{W}]\left[\pi_2^T-\underline{U}_2\right]-\beta_2[W^{\mathrm{S}}-\underline{W}]\left[\pi_1^T-\underline{U}_1\right]\right\}\leq 0,\ f\cdot\frac{\partial L}{\partial f}=0,$$

$$\frac{\partial L}{\partial \delta}=D\cdot\left\{(1-\beta_1-\beta_2)\frac{\partial W^{\mathrm{S}}}{\partial \delta}\left[\pi_1^T-\underline{U}_1\right]\left[\pi_2^T-\underline{U}_2\right]+\beta_1\frac{\partial \pi_1^T}{\partial \delta}\left[W^{\mathrm{S}}-\underline{W}\right]\left[\pi_2^T-\underline{U}_2\right]+\beta_2\frac{\partial \pi_2^T}{\partial \delta}\left[W^{\mathrm{S}}-\underline{W}\right]\left[\pi_1^T-\underline{U}_1\right]\right\}\leq 0,\ \delta\cdot\frac{\partial L}{\partial \delta}=0.$$

Since $\frac{\partial W^{\mathrm{S}}}{\partial r}=-2\delta<0$ and $\frac{\partial W^{\mathrm{S}}}{\partial f}=-2<0$, we have $\frac{\partial L}{\partial r}<0$ and $\frac{\partial L}{\partial f}<0$. These imply $r^s=0$ and $f^s=0$, and hence the subsidy is the best choice for all three parties. Moreover, we have $\frac{\partial \pi_1^T}{\partial \delta}=a-c_1-r-2(1+b)\delta<0$, $\frac{\partial \pi_2^T}{\partial \delta}=a-c_2-r-2(1+b)\delta<0$ by $\delta\in(\delta_2, \frac{a}{1+b})$, and $\frac{\partial W^{\mathrm{S}}}{\partial \delta}=2(a-r)-c_1-c_2-2(1+b)\delta>(\leq)\,0$ iff $\delta<(\geq)\frac{2(a-r)-c_1-c_2}{2(1+b)}$. Note that

$\delta_2 < \frac{2(a-r)-c_1-c_2}{2(1+b)} < \frac{a}{1+b}$.[23] Thus, there are two situations as follows.

First, if $\delta \in \left[\frac{2(a-r)-c_1-c_2}{2(1+b)}, \frac{a}{1+b}\right)$, then $\frac{\partial L}{\partial \delta} < 0$ by $\frac{\partial W^{S}}{\partial \delta} \le 0$, $\frac{\partial \pi_1^T}{\partial \delta} < 0$ and $\frac{\partial \pi_2^T}{\partial \delta} < 0$. These imply $\delta^s = 0$, which contradicts $\delta \ge \frac{2(a-r)-c_1-c_2}{2(1+b)}$. Second, if $\delta \in \left(\delta_2, \frac{2(a-r)-c_1-c_2}{2(1+b)}\right)$, then $\frac{\partial W^{S}}{\partial \delta} > 0$, $\pi_2^T = \left[a-(1+b)\delta - c_2 - r\right]\delta - f > 0$ iff $\delta < \frac{a-r-c_2}{1+b}$ with $\frac{a-r-c_2}{1+b} < \frac{2(a-r)-c_1-c_2}{2(1+b)}$, $\pi_2^T = \frac{(a-c_1)\left[a+(1+b)c_1-(2+b)c_2\right]}{(2+b)^2} > 0$ iff $c_2 < \frac{a+(1+b)c_1}{2+b}$ at $\delta = \delta_2$, and $\frac{\partial L}{\partial \delta} = D \cdot K$, where

$$K = \left\{ \begin{array}{l} (1-\beta_1-\beta_2)\left[2(a-r)-c_1-c_2-2(1+b)\delta\right]\left[\pi_1^T-\underline{U}_1\right]\left[\pi_2^T-\underline{U}_2\right] \\ +\beta_1\left[a-c_1-r-2(1+b)\delta\right]\left[W^{S}-\underline{W}\right]\left[\pi_2^T-\underline{U}_2\right]+\beta_2\left[a-c_2-r-2(1+b)\delta\right]\left[W^{S}-\underline{W}\right]\left[\pi_1^T-\underline{U}_1\right] \end{array} \right\}.$$

Some calculations show

$K = \beta_1\left[a-c_1-r-2(1+b)\delta\right]\left[W^{S}-\underline{W}\right]\left[\pi_2^T-\underline{U}_2\right]+\beta_2\left[a-c_2-r-2(1+b)\delta\right]\left[W^{S}-\underline{W}\right]\left[\pi_1^T-\underline{U}_1\right] < 0$ at $\delta = \frac{2(a-r)-c_1-c_2}{2(1+b)}$. Since the sign of K depends on the relative values of the three terms: $(1-\beta_1-\beta_2)\left[2(a-r)-c_1-c_2-2(1+b)\delta\right]\left[\pi_1^T-\underline{U}_1\right]\left[\pi_2^T-\underline{U}_2\right]$, $\beta_1\left[a-c_1-r-2(1+b)\delta\right]\left[W^{S}-\underline{W}\right]\left[\pi_2^T-\underline{U}_2\right]$ and $\beta_2\left[a-c_2-r-2(1+b)\delta\right]\left[W^{S}-\underline{W}\right]\left[\pi_1^T-\underline{U}_1\right]$, the sign of $\frac{\partial K}{\partial \delta}$ is uncertain, and so is the sign of $\frac{\partial L}{\partial \delta}$. However, if an optimal δ^s exists, then it must satisfy $\delta^s \in \left(\delta_2, \frac{2(a-r)-c_1-c_2}{2(1+b)}\right)$ at $\delta = \delta^s$. Moreover, to ensure $\pi_2^T \ge 0$ at $\delta = \delta^s$, the conditions $c_2 < \frac{a+(1+b)c_1}{2+b}$ and $\delta < \frac{a-r-c_2}{1+b}$ are needed. That is, if the model's parameters $(\beta_1, \beta_2, b, c_1, c_2, \underline{U}_1, \underline{U}_2, \underline{W})$ satisfy $c_2 < \frac{a+(1+b)c_1}{2+b}$ and the condition J with

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[23]We have $\frac{a}{1+b} - \frac{2(a-r)-c_1-c_2}{2(1+b)} = \frac{2r+c_1+c_2}{2(1+b)} > 0$ and $\delta_2 - \frac{2(a-r)-c_1-c_2}{2(1+b)} = \frac{[-2a+2r-c_1b+c_2(2+b)]}{2(1+b)(2+b)}$ $< \frac{[-2a+2\overline{r}-c_1b+c_2(2+b)]}{2(1+b)(2+b)} = \frac{-b^2(c_2-c_1)}{2(1+b)(2-b)(2+b)} < 0$.

$$J=\left\{\begin{array}{l}\frac{a-c_1}{2+b}<\delta^s<\frac{a-c_2}{(1+b)},\ \text{and}\\ \left\{\begin{array}{l}(1-\beta_1-\beta_2)\left\{\begin{array}{l}-2(1+b)\left[\pi_1^T-\underline{U}_1\right]\left[\pi_2^T-\underline{U}_2\right]+\left[2a-c_1-c_2-2(1+b)\delta^s\right]\left[a-c_1-2(1+b)\delta^s\right]\left[\pi_2^T-\underline{U}_2\right]\\+\left[2a-c_1-c_2-2(1+b)\delta^s\right]\left[a-c_2-2(1+b)\delta^s\right]\left[\pi_1^T-\underline{U}_1\right]\end{array}\right\}\\+\beta_1\left\{\begin{array}{l}\left[a-c_1-2(1+b)\delta^s\right]\left[2(a-r)-c_1-c_2-2(1+b)\delta^s\right]\left[\pi_2^T-\underline{U}_2\right]\\+\left[a-c_1-2(1+b)\delta^s\right]\left[W^S-\underline{W}\right]\left[a-c_2-2(1+b)\delta^s\right]-2(1+b)\left[W^S-\underline{W}\right]\left[\pi_2^T-\underline{U}_2\right]\end{array}\right\}\\+\beta_2\left\{\begin{array}{l}-2(1+b)\delta^s\left[W^S-\underline{W}\right]\left[\pi_1^T-\underline{U}_1\right]+\left[a-c_2-2(1+b)\delta^s\right]\left[2a-c_1-c_2-2(1+b)\delta^s\right]\left[\pi_1^T-\underline{U}_1\right]\\+\left[a-c_2-2(1+b)\delta^s\right]\left[W^S-\underline{W}\right]\left[a-c_1-2(1+b)\delta^s\right]\end{array}\right\}\end{array}\right\}<0\end{array}\right\}, \tag{A52}$$

then the optimal contract is $r^s=0$, $f^s=0$ and

$$\delta^{s2}\in\arg\left\{\left\{\begin{array}{l}(1-\beta_1-\beta_2)\left[2a-c_1-c_2-2(1+b)\delta\right]\left[\pi_1^T-\underline{U}_1\right]\left[\pi_2^T-\underline{U}_2\right]\\+\beta_1\left[a-c_1-2(1+b)\delta\right]\left[W^S-\underline{W}\right]\left[\pi_2^T-\underline{U}_2\right]\\+\beta_2\left[a-c_2-2(1+b)\delta\right]\left[W^S-\underline{W}\right]\left[\pi_1^T-\underline{U}_1\right]\end{array}\right\}=0\right\}. \tag{A53}$$

In summary, there are two equilibrium solutions: one is given in (A50) under the condition (A49), and the second one is given in (A53) under the condition (A52). These confirm Proposition 6.