Chapter 3

Asymptotic Equipartition Property

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Chapter Outline

Chap. 3 Asymptotic Equipartition Property

- **3.1** Asymptotic Equipartition Property Theorem
- **3.2** Consequences of the AEP: Data Compression
- **3.3** High-Probability Sets and Typical Set

3.1 Asymptotic Equipartition Property Theorem

Definition of convergence

Given a sequence of random variables, $X_1, X_2, \ldots$ we say that the sequence $X_1, X_2, \ldots$ converges to a random variable X

- **In probability** if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr\{|X_n - X| > \epsilon\} = 0$$

or, equivalently,

$$\lim_{n\to\infty} \Pr\{|X_n - X| < \epsilon\} = 1$$

Definition of convergence

- **In mean square** if

$$\lim_{n\to\infty} E[|X_n - X|^2] = 0$$

- **With probability 1** or called **almost surely** if

$$\Pr\left\{\lim_{n\to\infty} X_n = X\right\} = 1$$

Weak law of large numbers

For i.i.d. random variables $X_1, X_2, \ldots, X_n$ with common mean m, we have

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to m \quad \text{in probability.}$$

That is, for any $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr\left\{\left|\frac{1}{n}\sum_{i=1}^{n} X_i - m\right| > \epsilon\right\} = 0$$

AEP

Theorem 3.1.1 (AEP) If $X_1, X_2, \ldots$ are i.i.d. $\sim p(x)$, then

$$-\frac{1}{n}\log p(X_1, X_2, \ldots, X_n) \to H(X) \quad \text{in probability}$$

Proof. Let $Z_i = -\log p(X_i)$ be i.i.d. random variables. That is, $Z_i = -\log p[X_i = x]$ if $X_i = x$, we have

$$E[Z_i] = -\sum p[X_i = x]\log p[X_i = x] = H(X_i) = H(X)$$

Now, by the weak law of large numbers,

$$\frac{1}{n}\sum_i Z_i \to H(X) \quad \text{in probability}$$

$$\Rightarrow -\frac{1}{n}\sum_i \log p(X_i) \to H(X) \quad \text{in probability}$$

$$\Rightarrow -\frac{1}{n}\log p(X_1, X_2, \ldots, X_n) \to H(X) \quad \text{in probability} \quad \square$$

Interpretation of AEP

- When n is sufficient large, $p(X_1, X_2, \ldots, X_n) = 2^{-nH(X)}$ with high probability.
- For example, Let the random number X_i with probability $P[X_i = 1] = p$ and $P[X_i = 0] = 1 - p = q$. If $X_1, X_2, \ldots, X_n$ are i.i.d.,

$$p(X_1, X_2, \ldots, X_n) = p^{\sum X_i} q^{n - \sum X_i}.$$

When $n \to \infty$,

$$p(X_1, X_2, \ldots, X_n) \to p^{np} q^{nq} = 2^{-nH}.$$

It means that the number of 1's in the sequence is close to np, and all such sequences have roughly the same probability 2^{-nH}.

Interpretation of AEP

- Thus for large n we can divide the sequences $X_1, X_2, \ldots, X_n$ into two types: the typical type consisting of sequences each with probability roughly 2^{-nH}, and another type, consisting of other sequences.

Typical set

Definition (Typical set) The typical set $A_\epsilon^{(n)}$ with respect to $p(x)$ is the set of sequence $(x_1, x_2, \ldots, x_n) \in \mathcal{X}^n$ with the property

$$2^{-n(H(X)+\epsilon)} \le p(x_1, x_2, \ldots, x_n) \le 2^{-n(H(X)-\epsilon)}.$$

Theorem 3.1.2

1. If $(x_1, x_2, \ldots, x_n) \in A_\epsilon^{(n)}$, then

$$H(X) - \epsilon \le -\frac{1}{n} \log p(x_1, x_2, \ldots, x_n) \le H(X) + \epsilon$$

.

Proof. By the definition of typical set. $\square$.

Theorems

Theorem 3.1.2

2. $\Pr\{A_\epsilon^{(n)}\} > 1-\epsilon$ for n sufficiently large.

Proof. This property follows directly from Theorem 3.1.1, since the convergence in the mean can be written as

$$\Pr\left\{\left|-\frac{1}{n}\log p(X_1, X_2, \ldots, X_n) - H(X)\right| < \epsilon\right\} > 1-\delta$$

Setting $\delta = \epsilon$, we obtain the desired result. $\square$

Theorems

Theorem 3.1.2

3. $|A_\epsilon^{(n)}| \le 2^{n(H(X)+\epsilon)}$, where $|A|$ denotes the number of elements in the set A.

Proof.

$$1 = \sum_{\mathbf{x}\in\mathcal{X}^n} p(\mathbf{x}) \ge \sum_{\mathbf{x}\in A_\epsilon^{(n)}} p(\mathbf{x})$$

$$\ge \sum_{\mathbf{x}\in A_\epsilon^{(n)}} 2^{-n[H(X)+\epsilon]} = 2^{-n[H(X)+\epsilon]}|A_\epsilon^{(n)}| \quad \Box$$

Theorems

Theorem 3.1.2

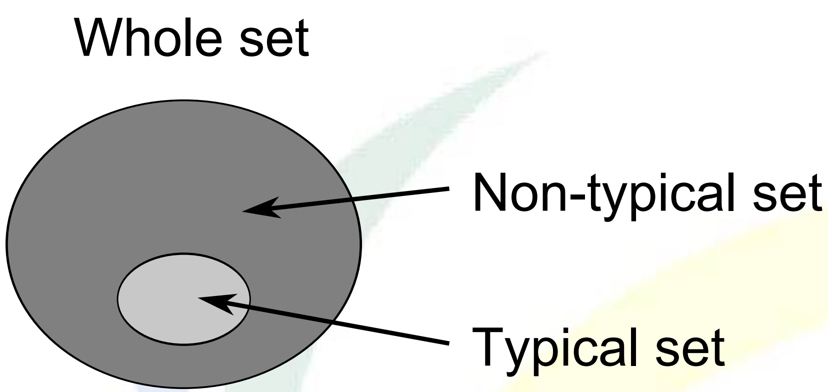
4. $|A_\epsilon^{(n)}| \geq (1-\epsilon)2^{n(H(X)-\epsilon)}$ for n sufficiently large.

Proof. For n sufficiently large, $\Pr\{A_\epsilon^{(n)}\} > 1-\epsilon$, so that,

$$1-\epsilon < \Pr\{A_\epsilon^{(n)}\} \leq \sum_{\mathbf{x}\in A_\epsilon^{(n)}} 2^{-n[H(X)-\epsilon]} = 2^{-n[H(X)-\epsilon]}|A_\epsilon^{(n)}| \quad \Box$$

3.2 Consequences of the AEP: Data Compression

Typical set and source coding



- There are $|\mathcal{X}|^n$ elements in the whole set.
- There are $|A_\epsilon^{(n)}| \approx 2^{n(H+\epsilon)}$ elements in the typical set. We need $n(H+\epsilon)+1$ bits to encode these elements, and one addition bit to indicate they are typical sequences.
- There are $|\mathcal{X}|^n - |A_\epsilon^{(n)}|$ elements in the nontypical set. We can use $n\log|\mathcal{X}|+1$ bits to encode them, and one addition bit to indicate they are non-typical sequences.

Average length of codeword

$$
\begin{aligned}
E[l(X^n)] &= \sum_{x^n} p(x^n) l(x^n) \\
&= \sum_{x^n \in A_\epsilon^{(n)}} p(x^n) l(x^n) + \sum_{x^n \in [A_\epsilon^{(n)}]^c} p(x^n) l(x^n) \\
&\le \sum_{x^n \in A_\epsilon^{(n)}} p(x^n)[n(H+\epsilon)+2] \\
&+ \sum_{x^n \in [A_\epsilon^{(n)}]^c} p(x^n)[n \log|\mathcal{X}| + 2] \\
&= \Pr\{A_\epsilon^{(n)}\}[n(H+\epsilon)+2] + \Pr\{[A_\epsilon^{(n)}]^c\}[n\log|\mathcal{X}|+2] \\
&\le n(H+\epsilon) + \epsilon n \log|\mathcal{X}| + 2 \\
&= n(H+\epsilon')
\end{aligned}
$$

where $\epsilon' = \epsilon + \epsilon \log|\mathcal{X}| + \frac{2}{n}$

Theorems

Theorem 3.2.1 Let X^n be i.i.d. $\sim p(x)$. Let $\epsilon > 0$. Then there exists a code that maps sequences x^n of length n into binary strings such that the mapping is one-to-one (and therefore invertible) and

$$E\left[\frac{1}{n}l(X^n)\right] \leq H(X) + \epsilon$$

for n sufficiently large.