# Chapter 7

# *Channel Capacity*

Peng-Hua Wang

Graduate Inst. of Comm. Engineering

National Taipei University

# Chapter Outline

**Chap. 7** Channel Capacity

- **7.1** Examples of Channel Capacity
- **7.2** Symmetric Channels
- **7.3** Properties of Channel Capacity
- **7.4** Preview of the Channel Coding Theorem
- **7.5** Definitions
- **7.6** Jointly Typical Sequences
- **7.7** Channel Coding Theorem
- **7.8** Zero-Error Codes
- **7.9** Fano's Inequality and the Converse to the Coding Theorem

# 7.1 Examples of Channel Capacity

# Channel Model

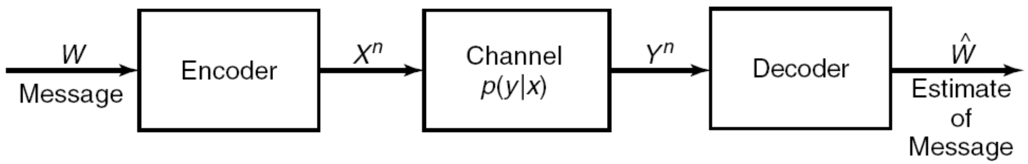



**FIGURE 7.1.** Communication system.

- Operational channel capacity: the bit number to represent the maximum number of distinguishable signals for $n$ uses of a communication channel.
  - In $n$ transmission, we can send $M$ signals without error, the channel capacity is $\log M/n$ bits per transmission.
- Information channel capacity: the maximum mutual information
- Operational channel capacity is equal to Information channel capacity.
  - Fundamental theory and central success of information theory.

# Channel capacity

**Definition 1 (Discrete Channel)** *A system consisting of an input alphabet* $\mathcal{X}$ *and output alphabet* $\mathcal{Y}$ *and a probability transition matrix* $p(y|x)$.

**Definition 2 (Channel capacity)** *The "information" channel capacity of a discrete memoryless channel is*

$$C = \max_{p(x)} I(X;Y)$$

*where the maximum is taken over all possible input distribution* $p(x)$.

- Operational definition of channel capacity: The highest rate in bits per channel use at which information can be sent.
- Shannon's second theorem: The information channel capacity is equal to the operational channel capacity.

# Example 1

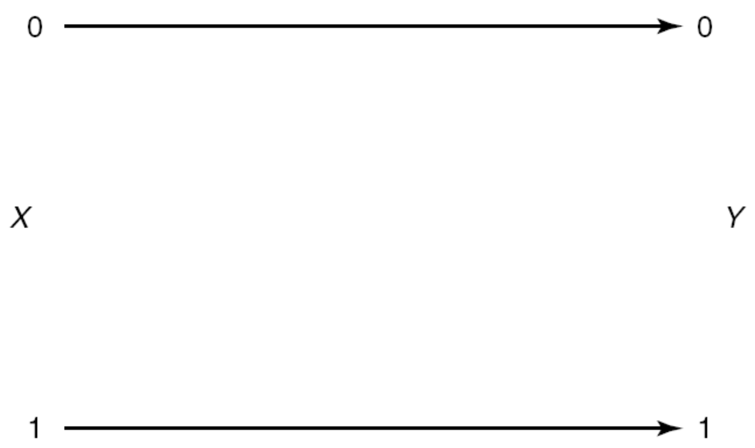



**FIGURE 7.2.** Noiseless binary channel. $C = 1$ bit.

Noiseless binary channel

$$p(Y=0)=p(X=0)=\pi_0, \quad p(Y=1)=p(X=1)=\pi_1=1-\pi_0$$

$$\begin{aligned} I(X;Y) &= H(Y)-H(Y|X)=H(Y) \\ &\leq 1 \\ \text{“}={}\text{”} &\Rightarrow \pi_0=\pi_1=1/2 \end{aligned}$$

# Example 2

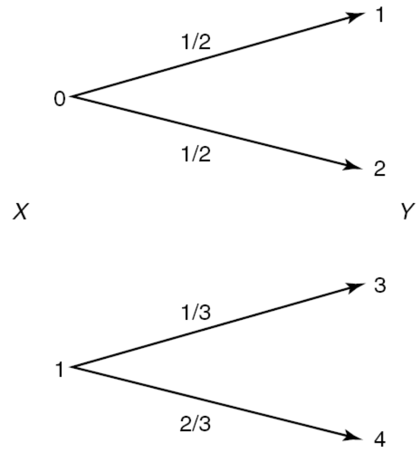



**FIGURE 7.3.** Noisy channel with nonoverlapping outputs. $C = 1$ bit.

Noisy channel with non-overlapping outputs

$$p(X=0) = \pi_0, \quad p(X=1) = \pi_1 = 1 - \pi_0$$

$$p(Y=1) = \pi_0 p, \quad p(Y=2) = \pi_0(1-p), \quad p = 1/2$$

$$p(Y=3) = \pi_1 q, \quad p(Y=4) = \pi_1(1-q), \quad q = 1/3$$

$$I(X;Y) = H(Y) - H(Y|X) = H(Y) - \pi_0 H(p) - \pi_1 H(q)$$

$$= H(\pi_0) = H(X) \leq 1$$

# Noisy Typewriter

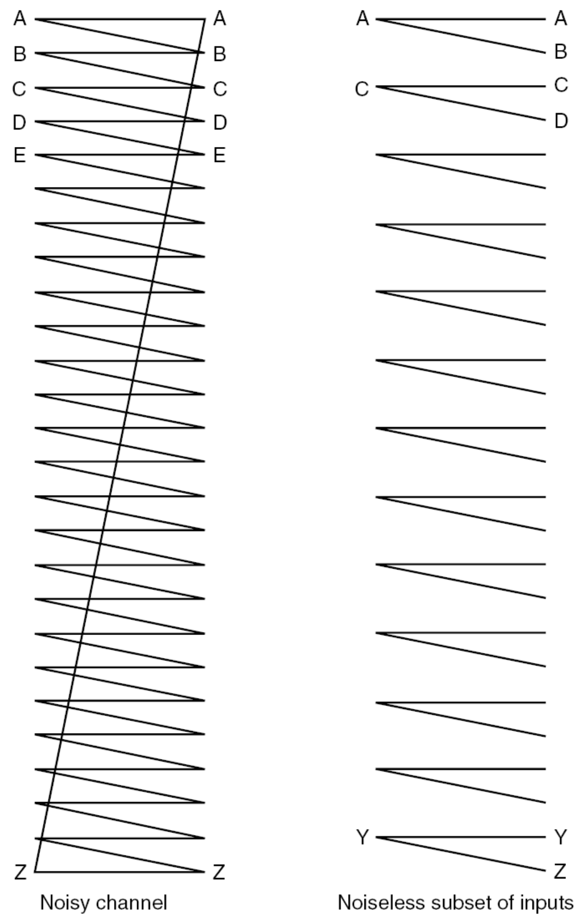



**FIGURE 7.4.** Noisy Typewriter. $C = \log 13$ bits.

Noisy typewriter

# Noisy Typewriter

$$
\begin{aligned}
& I(X;Y) \\
&= H(Y) - H(Y|X) \\
&= H(Y) - \sum_x p(x) H(Y|X=x) \\
&= H(Y) - \sum_x p(x) H(\tfrac{1}{2}) \\
&= H(Y) - H(\tfrac{1}{2}) \\
&\leq \log 26 - 1 = \log 13 \\
C &= \max I(X;Y) = \log 13
\end{aligned}
$$

# Binary Symmetric Channel (BSC)

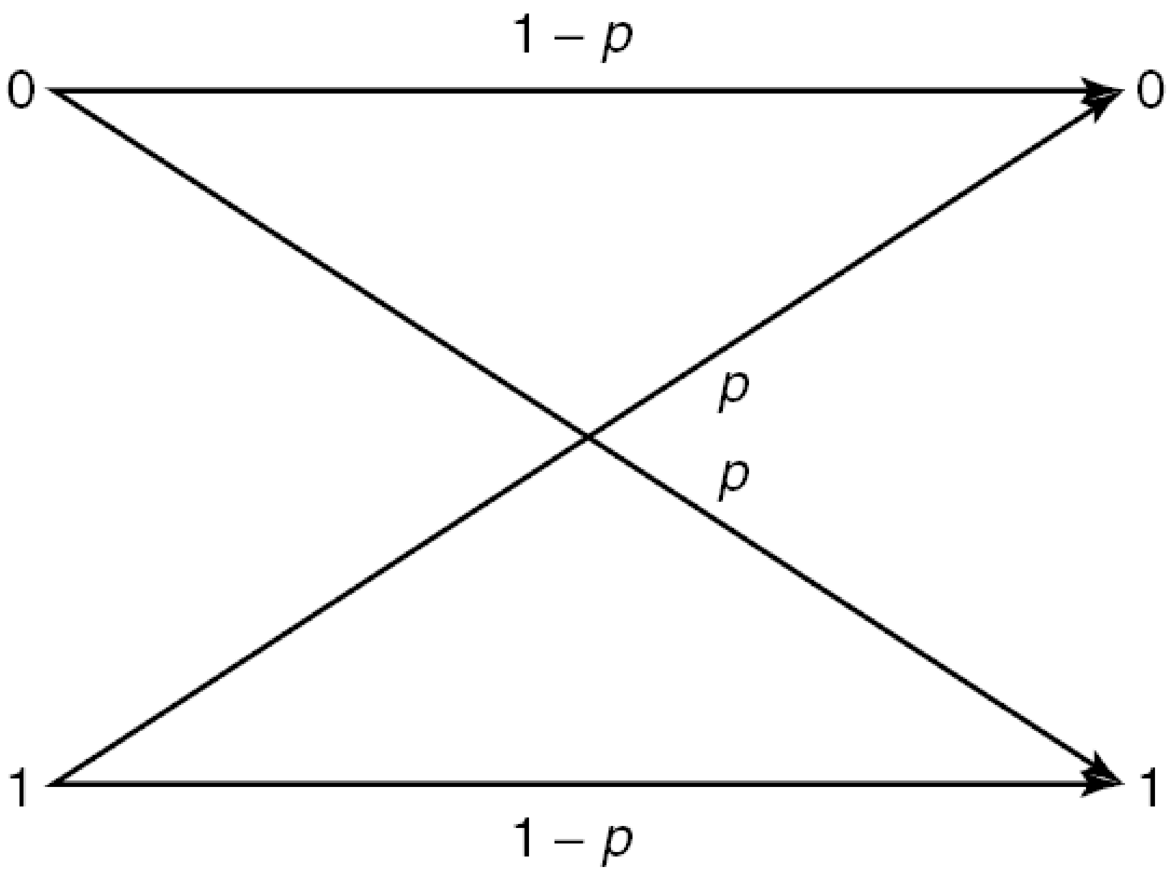



**FIGURE 7.5.** Binary symmetric channel. $C = 1 - H(p)$ bits.

# Binary Symmetric Channel (BSC)

$$
\begin{aligned}
& I(X;Y) \\
&= H(Y) - H(Y|X) \\
&= H(Y) - \sum_x p(x) H(Y|X=x) \\
&= H(Y) - \sum_x p(x) H(p) \\
&= H(Y) - H(p) \\
&\leq 1 - H(p)
\end{aligned}
$$

$$C = \max I(X;Y) = 1 - H(p)$$

# Binary Erasure Channel

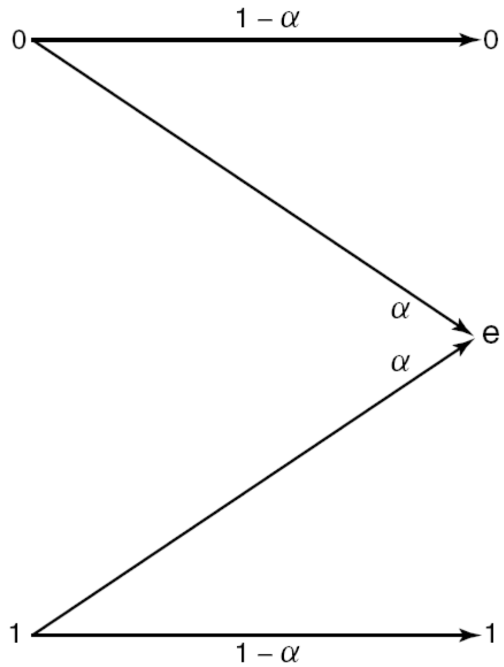



**FIGURE 7.6.** Binary erasure channel.

# Binary Erasure Channel

$$
\begin{aligned}
& I(X;Y) \\
&= H(Y) - H(Y|X) \\
&= H(Y) - \sum_x p(x)H(Y|X=x) \\
&= H(Y) - \sum_x p(x)H(\alpha) \\
&= H(Y) - H(\alpha)
\end{aligned}
$$

$$H(Y) = (1-\alpha)H(\pi_0) + H(\alpha)$$

$$C = \max I(X;Y) = 1 - \alpha$$

# 7.3 Properties of Channel Capacity

# Properties of Channel Capacity

- $C \geq 0$.
- $C \leq \log |\mathcal{X}|$.
- $C \leq \log |\mathcal{Y}|$.
- $I(X;Y)$ is a continuous function of $p(x)$,
- $I(X;Y)$ is a concave function of $p(x)$,

# 7.4 Preview of the Channel Coding Theorem

# Preview of the Channel Coding Theorem

- For each input $n$-sequence, there are approximately $2^{nH(Y|X)}$, possible $Y$ sequences.
- The total number of possible (typical) $Y$ sequences is $2^{nH(Y)}$.
- This set has to be divided into sets of size $2^{nH(Y|X)}$ corresponding to the different input $X$ sequences.
- The total number of disjoint sets is less than or equal to $2^{nH(Y)}/2^{nH(Y|X)} = 2^{n(H(Y)-H(Y|X))} = 2^{nI(X;Y)}$
- We can send at most $2^{nI(X;Y)}$ distinguishable sequences of length $n$.

# Example

X: 2^(nH(X)) = 6

Y: 2^(nH(Y)) = 4

| | 001100 | 101101 | 101000 | 101110 | 011101 | 011001 |
|---|---|---|---|---|---|---|
| 010101 | | A | | B | | C |
| 010100 | D | | E | | F | |
| 101010 | | G | | H | | I |
| 101011 | J | | K | | L | |

jointly typical sequences: 2^(nH(X,Y)) = 12

- $6$ typical sequences for $X^n$. $4$ typical sequences for $Y^n$.
- $12$ typical sequences for $(X^n, Y^n)$.
- For every $X^n$, we have

$$2^{nH(X,Y)}/2^{nH(X)} = 2^{nH(Y|X)} = 2 \quad \text{typical } Y^n.$$

e.g., for $X^n = 001100 \Rightarrow Y^n = 010100, 101011.$

# Example

- Since we have $2^{nH(Y)} = 4$ typical $Y^n$ in total, how many typical $X^n$ can these typical $Y^n$ be assigned?

$$2^{nH(Y)}/2^{nH(Y|X)} = 2^{n(H(Y)-H(Y|X))} = 2^{nI(X;Y)} = 2.$$

- Can we assign more typical $X^n$? No. For some $Y^n$ received, we can't not determine which $X^n$ is received. e.g., If we use 001100, 101101, and 101000 as codewords, we can't determine which codeword is sent when we receive 101011.

# 7.5 Definitions

# Communication Channel

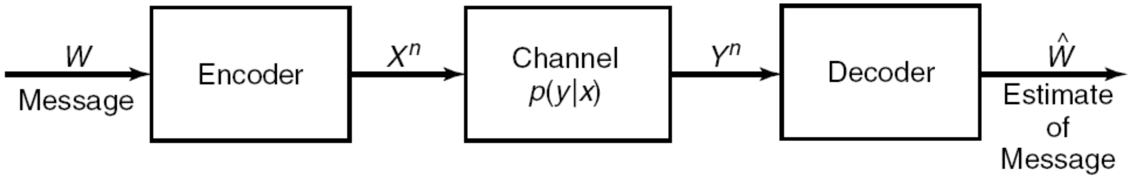



**FIGURE 7.8.** Communication channel.

- Message $W \in \{1, 2, ..., M\}$.
- Encoder: input $W$, output $X^n \equiv X^n(W) \in \mathcal{X}^n$
  - $n$ is the length of the signal. We then transmit the signal via the channel by using the channel $n$ times. Every time we send a symbol of the signal.
- Channel: input $X^n$, output $Y^n$ with distribution $p(y^n|x^n)$
- Decoder: input $Y^n$, output $\hat{W} = g(Y^n)$ where $g(Y^n)$ is a deterministic decoding rule.
- If $\hat{W} \neq W$, an error occurs.

# Definitions

**Definition 3 (Discrete Channel)** *A discrete channel, denoted by* $(\mathcal{X}, p(y|x), \mathcal{Y})$*, consists of two finite sets* $\mathcal{X}$ *and* $\mathcal{Y}$ *and a collection of probability mass functions* $p(y|x)$.

- $X$: input, $Y$ :output, for every input $x \in \mathcal{X}$, $\sum_y p(y|x) = 1$.

**Definition 4 (Discrete Memoryless Channel, DMC)** *The* $n$*th extension of the discrete memoryless channel is the channel* $(\mathcal{X}^n, p(y^n|x^n), \mathcal{Y}^n)$ *where*
$p(y_k|x^k, y^{k-1}) = p(y_k|x_k), k = 1, 2, \ldots, n.$

- Without feedback: $p(x_k|x^{k-1}, y^{k-1}) = p(x_k|x^{k-1})$
- $n$th extension of DMC without feedback:

$$p(y^n|x^n) = \prod_{i=1}^{n} p(y_i|x_i).$$

# Definitions

**Definition 5** $(M, n)$ *code] An* $(M, n)$ *code for the channel* $(\mathcal{X}, p(y|x), \mathcal{Y})$ *consists of the following:*

1. *An index set* $\{1, 2, \ldots, M\}$.
2. *An encoding function* $X^n : \{1, 2, \ldots, M\} \to \mathcal{X}^n$. *The codewords are* $x^n(1), x^n(2), \ldots, x^n(M)$. *The set of codewords is called the* **codebook**.
3. *A decoding function* $g : \mathcal{Y}^n \to \{1, 2, \ldots, M\}$

# Definitions

**Definition 6 (Conditional probability of error)**

$$\lambda_i = \Pr(g(Y^n) \neq i | X^n = x^n(i)) = \sum_{g(y^n) \neq i} p(y^n | x^n(i))$$

$$= \sum_{y^n} p(y^n | x^n(i)) I(g(y^n) \neq i)$$

- $I(\cdot)$ is the indicator function.

# Definitions

**Definition 7 (Maximal probability of error)**

$$\lambda^{(n)} = \max_{i \in \{1,2,\ldots,M\}} \lambda_i$$

**Definition 8 (Average probability of error)**

$$P_e^{(n)} = \frac{1}{M} \sum_{i=1}^{M} \lambda_i$$

- The decoding error is

$$\Pr(g(Y^n) \neq W) = \sum_{i=1}^{M} \Pr(W = i) \Pr(g(Y^n) \neq i | W = i)$$

If the index $W$ is chosen uniformly from $\{1, 2, \ldots, M\}$, then $P_e^{(n)} = \Pr(g(Y^n) \neq W)$.

# Definitions

**Definition 9 (Rate)** *The rate $R$ of an $(M, n)$ code is*

$$R = \frac{\log M}{n} \quad \textit{bits per transmission}$$

**Definition 10 (Achievable rate)** *A rate $R$ is said to be achievable if there exists a $(\lceil 2^{nR} \rceil, n)$ code such that the maximal probability of error $\lambda^{(n)}$ tends to $0$ as $n \to \infty$.*

**Definition 11 (Channel capacity)** *The capacity of a channel is the supremum of all achievable rates.*

# 7.6 Jointly Typical Sequences

# Definitions

**Definition 12 (Jointly typical sequences)** *The set* $A_\epsilon^{(n)}$ *of jointly typical sequences* $\{(x^n, y^n)\}$ *with respect to the distribution* $p(x,y)$ *is defined by*

$$A_\epsilon^{(n)} = \left\{ (x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n : \left| -\frac{1}{n} \log p(x^n) - H(X) \right| < \epsilon, \right.$$
$$\left| -\frac{1}{n} \log p(y^n) - H(Y) \right| < \epsilon,$$
$$\left. \left| -\frac{1}{n} \log p(x^n, y^n) - H(X,Y) \right| < \epsilon \right\}$$

*where*

$$p(x^n, y^n) = \prod_{i=1}^{n} p(x_i, y_i)$$

# Joint AEP

**Theorem 1 (Joint AEP)** *Let* $(X^n, Y^n)$ *be sequences of length* $n$ *drawn i.i.d. according to* $p(x^n, y^n)$*. Then:*

*1.* $\Pr\left((x^n, y^n) \in A_\epsilon^{(n)}\right) \to 1$ *as* $n \to \infty$*.*

*2.* $\left|A_\epsilon^{(n)}\right| \leq 2^{n(H(X,Y)+\epsilon)}$

*3. If* $(\tilde{X}^n, \tilde{Y}^n) \sim p(x^n)p(y^n)$ *[i.e.,* $\tilde{X}^n$ *and* $\tilde{Y}^n$ *are independent with the same marginals], then*

$$\Pr\left((\tilde{X}^n, \tilde{Y}^n) \in A_\epsilon^{(n)}\right) \leq 2^{-n(I(X;Y)-3\epsilon)}.$$

*Also, for sufficient large* $n$*,*

$$\Pr\left((\tilde{X}^n, \tilde{Y}^n) \in A_\epsilon^{(n)}\right) \geq (1-\epsilon)2^{-n(I(X;Y)+3\epsilon)}.$$

# Joint AEP

**Theorem 2 (Joint AEP)** *1.* $\Pr\left((x^n, y^n) \in A_\epsilon^{(n)}\right) \to 1$ *as* $n \to \infty$.

**Proof.** Given $\epsilon > 0$, define events $A, B, C$ as

$$A \triangleq \left\{X^n : \left|-\frac{1}{n}\log p(X^n) - H(X)\right| \geq \epsilon\right\}$$

$$B \triangleq \left\{Y^n : \left|-\frac{1}{n}\log p(Y^n) - H(Y)\right| \geq \epsilon\right\}$$

$$C \triangleq \left\{(X^n, Y^n) : \left|-\frac{1}{n}\log p(X^n, Y^n) - H(X, Y)\right| \geq \epsilon\right\},$$

# Joint AEP

Then, by weak law of large number, there exists $n_1, n_2, n_3$ such that,

$$\Pr(A) < \frac{\epsilon}{3}, \quad \forall n > n_1, \quad \Pr(B) < \frac{\epsilon}{3}, \quad \forall n > n_2,$$
$$\Pr(C) < \frac{\epsilon}{3}, \quad \forall n > n_3.$$

Thus,

$$\begin{aligned}
&\Pr\left((x^n, y^n) \in A_\epsilon^{(n)}\right) = \Pr(A^c \cap B^c \cap C^c) \\
=&1 - \Pr(A \cup B \cup C) \geq 1 - (\Pr(A) + \Pr(B) + \Pr(C)) \\
\geq&1 - \epsilon
\end{aligned}$$

for all $n > \max\{n_1, n_2, n_3\}$. $\square$

# Joint AEP

**Theorem 3 (Joint AEP)** *2.* $\left|A_\epsilon^{(n)}\right| \leq 2^{n(H(X,Y)+\epsilon)}$

**Proof.**

$$1 = \sum p(x^n, y^n) \geq \sum_{(x^n,y^n)\in A_\epsilon^{(n)}} p(x^n, y^n)$$

$$\geq |A_\epsilon^{(n)}| 2^{-n(H(X,Y)+\epsilon)}$$

Thus,

$$\left|A_\epsilon^{(n)}\right| \leq 2^{n(H(X,Y)+\epsilon)}. \quad \square$$

# Joint AEP

**Theorem 4 (Joint AEP)** *3. If $(\tilde{X}^n, \tilde{Y}^n) \sim p(x^n)p(y^n)$ [i.e., $\tilde{X}^n$ and $\tilde{Y}^n$ are independent with the same marginals], then*

$$\Pr\left((\tilde{X}^n, \tilde{Y}^n) \in A_\epsilon^{(n)}\right) \leq 2^{-n(I(X;Y)-3\epsilon)}.$$

*Also, for sufficient large $n$,*

$$\Pr\left((\tilde{X}^n, \tilde{Y}^n) \in A_\epsilon^{(n)}\right) \geq (1-\epsilon)2^{-n(I(X;Y)+3\epsilon)}.$$

# Joint AEP

**Proof.**

$$\Pr\left((\tilde{X}^n, \tilde{Y}^n) \in A_\epsilon^{(n)}\right) = \sum_{(x^n,y^n)\in A_\epsilon^{(n)}} p(x^n)p(y^n)$$

$$\leq 2^{n(H(X,Y)+\epsilon)} 2^{-n(H(X)-\epsilon)} 2^{-n(H(Y)-\epsilon)}$$

$$= 2^{-n(I(X;Y)-3\epsilon)}.$$

For sufficient large $n$, $\Pr\left(A_\epsilon^{(n)}\right) \geq 1-\epsilon$, and therefore

$$1-\epsilon \leq \sum_{(x^n,y^n)\in A_\epsilon^{(n)}} p(x^n, y^n) \leq \left|A_\epsilon^{(n)}\right| 2^{-n(H(X,Y)-\epsilon)}.$$

and

$$\left|A_\epsilon^{(n)}\right| \geq (1-\epsilon) 2^{n(H(X,Y)-\epsilon)}$$

# Joint AEP

$$
\begin{aligned}
&\Pr\left((\tilde{X}^n, \tilde{Y}^n) \in A_\epsilon^{(n)}\right) \\
=&\sum_{(x^n,y^n)\in A_\epsilon^{(n)}} p(x^n)p(y^n) \\
\geq&(1-\epsilon)2^{n(H(X,Y)-\epsilon)}2^{-n(H(X)+\epsilon)}2^{-n(H(Y)+\epsilon)} \\
=&(1-\epsilon)2^{-n(I(X;Y)+3\epsilon)}
\end{aligned}
$$

# Joint AEP: Conclusion

- There are about $2^{n(H(X)}$ typical $X$ sequences, and about $2^{n(H(Y)}$ typical $X$ sequences.
- There are about $2^{n(H(X,Y)}$ jointly typical sequences.
- Randomly chosen a pair of typical $X^n$ and typical $Y^n$, the probability that it is jointly typical is about $2^{-nI(X;Y)}$.

# 7.7 Channel Coding Theorem

# Channel Coding Theorem

**Theorem 5 (Channel coding theorem)** *For every rate $R < C$, there exists a sequence of $(2^{nR}, n)$ codes with maximum probability of error $\lambda^{(n)} \to 0$.*

*Conversely, any sequence of $(2^{nR}, n)$ codes with $\lambda^{(n)} \to 0$ must have $R \leq C$.*

- We have to prove two parts.
  - $R < C \to$ achievable.
  - Achievable $\to R \leq C$.
- Main ideas.
  - Random encoding (random code)
  - Jointly typical decoding

# Random Code

- Generate a $(2^{nR}, n)$ code at random according to the distribution $p(x)$ (fixed). That is, the $2^{nR}$ codewords have the distribution

$$p(x^n) = \prod_{i=1}^{n} p(x_i)$$

- A particular code $\mathcal{C}$ is the matrix with $2^{nR}$ codewords as the row.

$$\mathcal{C} = \begin{bmatrix} x_1(1) & x_2(1) & \cdots & x_n(1) \\ x_1(2) & x_2(2) & \cdots & x_n(2) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(2^{nR}) & x_2(2^{nR}) & \cdots & x_n(2^{nR}) \end{bmatrix}$$

- The code $\mathcal{C}$ is revealed to both sender and receiver. Both sender and receiver are also assumed to know the channel transition matrix $p(y|x)$ for the channel.

# Random Code

- There are $(|\mathcal{X}|^n)^{2^{nR}}$ different codes.
- The probability of a particular code $\mathcal{C}$ is

$$\Pr(\mathcal{C}) = \prod_{w=1}^{2^{nR}} \prod_{i=1}^{n} p\left(x_i(w)\right)$$

# Transmission and Channel

- A message $W$ is chosen according to a uniform distribution

$$\Pr[W = w] = 2^{-nR}, \quad w = 1, 2, \ldots, 2^{nR}.$$

- The $w$th codeword $X^n(w)$, corresponding to the $w$th row of $\mathcal{C}$, is sent over the channel.
- The receiver receives a sequence $Y^n$ according to the distribution

$$P(y^n|x^n(w)) = \prod_{i=1}^{n} p(y_i|x_i(w)).$$

That is, use the DMC channel for $n$ times.

# Jointly Typical Decoding

- The receiver declares that the message $\hat{W}$ was sent if
  - $\left(X^n(\hat{W}), Y^n\right)$ is jointly typical.
  - There is no other jointly typical pair for $Y^n$. That is, there is no other $W' \neq \hat{W}$ such that $W', Y^n$ is jointly typical.
- If no such $\hat{W}$ exists or if there is more than one such, an error is declared ($\hat{W} = 0$).
- There is decoding error if $\hat{W} \neq W$. Let $\mathcal{E}$ be the event $[\hat{W} \neq W]$.

# Proof of $R < C \to$ Achievable

- The average probability of error averaged over all codewords in the codebook, and averaged over all codebooks.

$$\Pr(\mathcal{E}) = \sum_{\mathcal{C}} P_e^{(n)}(\mathcal{C}) = \sum_{\mathcal{C}} \Pr(\mathcal{C}) \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \lambda_w(\mathcal{C})$$

$$= \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \sum_{\mathcal{C}} \Pr(\mathcal{C}) \lambda_w(\mathcal{C})$$

  - $P_e^{(n)}(\mathcal{C})$ is defined for jointly typical decoding.
  - By the symmetry of the code construction, $\sum_{\mathcal{C}} \Pr(\mathcal{C})\lambda_w(\mathcal{C})$ does not depend on $w$.

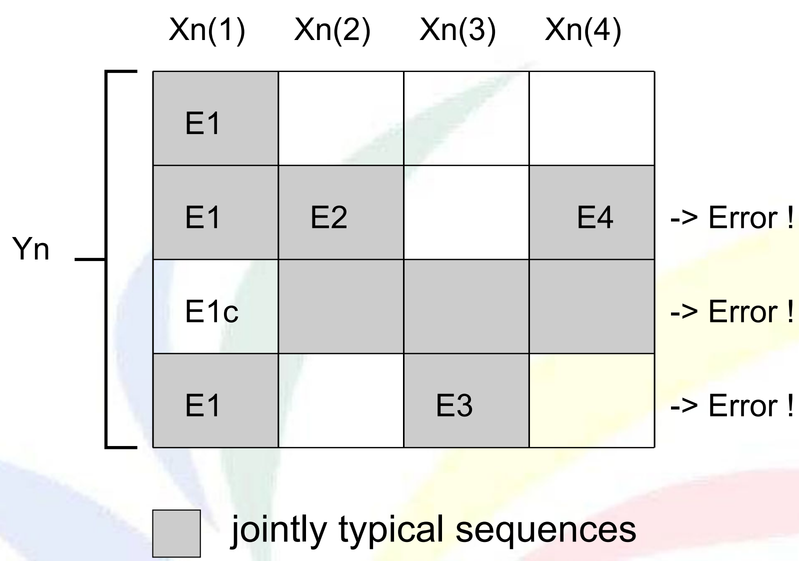
# Proof of $R < C \to$ Achievable

- Therefore,

$$\begin{aligned}
\Pr(\mathcal{E}) &= \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \sum_{\mathcal{C}} \Pr(\mathcal{C}) \lambda_w(\mathcal{C}) \\
&= \sum_{\mathcal{C}} \Pr(\mathcal{C}) \lambda_w(\mathcal{C}) \quad \text{for any } w \\
&= \sum_{\mathcal{C}} \Pr(\mathcal{C}) \lambda_1(\mathcal{C}) = \Pr(\mathcal{E} | W = 1)
\end{aligned}$$

- Define $E_i = \{(X^n(i), Y^n)$ is jointly typical pair$\}$ for $i = 1, 2, \ldots, 2^{nR}$ where $Y^n$ is the channel output when the first codeword $X^n(1)$ was sent. Then decoding error is declared if
  - $E_1^c$: The transmitted codeword and the received sequence are not jointly typical.
  - $E_2 \cup E_3 \cup \cdots \cup E_{2^{nR}}$: a wrong codeword is jointly typical with the received sequence.

# Proof of $R < C \rightarrow$ Achievable




- $Y^n$ is the channel output when the first codeword $X^n(1)$ was sent.
- $E_1^c$: The transmitted codeword and the received sequence are not jointly typical.
- $E_2, E_3, \ldots, E_{2^{nR}}$: wrong codewords that are jointly typical with the received sequence.

# Proof of $R < C \to$ Achievable

- The average error

$$\Pr(\mathcal{E}|W=1) = P(E_1^c \cup E_2 \cup E_3 \cup \cdots \cup E_{2^{nR}}|W=1)$$
$$\leq P(E_1^c|W=1) + \sum_{i=2}^{2^{nR}} P(E_i|W=1)$$

- By AEP,

$$P(E_1^c|W=1) \leq \epsilon \quad \text{for } n \text{ sufficiently large}$$
$$P(E_i|W=1) \leq 2^{-n(I(X;Y)-3\epsilon)}$$

($Y^n$ and $X^n(1)$ are jointly typical.)

# Proof of $R < C \to$ Achievable

- We have

$$\begin{aligned}\Pr(\mathcal{E}|W=1) &\leq \epsilon + (2^{nR}-1)2^{-n(I(X;Y)-3\epsilon)} \\ &\leq \epsilon + 2^{nR}2^{-n(I(X;Y)-3\epsilon)} \\ &= \epsilon + 2^{-n(I(X;Y)-R-3\epsilon)}\end{aligned}$$

- If $I(X;Y) - R - 3\epsilon > 0$, then $2^{-n(I(X;Y)-R-3\epsilon)} < \epsilon$ for $n$ sufficiently large, and

$$\Pr(\mathcal{E}|W=1) \leq 2\epsilon.$$

- So far, we prove that: for any $\epsilon$, if $R < I(X;Y)$ and $n$ sufficient large, the average decoding error $\Pr(\mathcal{E}) = \Pr(\mathcal{E}|W=1) < 2\epsilon$.
- What do we need? If $R < C$, the maximum error probability $\lambda^{(n)} \to 0$.

(We are almost there. Almost...)

# Proof of $R < C \to$ Achievable, final part

- Choose $p(x)$ such that $I(X;Y)$ is maximum. That is, choose $p(x)$ such that $I(X;Y)$ achieve channel capacity $C$. Then the condition $R < I(X;Y) - 3\epsilon$ can be replaced by the achievability condition $R < C - 3\epsilon$.
- Since the average probability of error over codebooks is less then $2\epsilon$, there exists at least one codebook $\mathcal{C}^*$ such that $\Pr(\mathcal{E}|\mathcal{C}^*) < 2\epsilon$.
  - $\mathcal{C}^*$ can be found by an exhaustive search over all codes.
- Since $W$ is chosen uniformly, we have

$$\Pr(\mathcal{E}|\mathcal{C}^*) = \frac{1}{2^{nR}} \sum_{i=1}^{2^{nR}} \lambda_i(C^*) \leq 2\epsilon$$

which implies that the maximal error probability of the better half codewords is less than $4\epsilon$.

  - There are 10 students. Their average score is 40. Then the highest

# Proof of $R < C \to$ Achievable, final part

- We throw away the worst half of the codewords in the best codebook $\mathcal{C}^*$. The new code has a maximal probability of error less than $4\epsilon$. However, we construct a $\left(2^{nR}/2, n\right)$ or $\left(2^{n(R-1/n)}, n\right)$ code. The rate of the new code is $R - 1/n$.
- Summary. If $R - 1/n < C - 3\epsilon$ for any $\epsilon$, then $\lambda^{(n)} \leq 4\epsilon$ for $n$ sufficiently large.

# 7.8 Zero-Error Codes

# No error $\to R \leq C$

- Assume that we have a $(2^{nR}, n)$ code with zero probability of error.
  - $W$ is determined by $Y^n$. $p(g(Y^n) = W) = 1$. $H(W|Y^n) = 0$.
- To obtain a strong bound, assume that $W$ is uniformly distributed over $\{1, 2, \ldots, 2^{nR}\}$.

$$\begin{aligned}
nR = H(W) &= H(W|Y^n) + I(W;Y^n) = I(W;Y^n) \\
&\leq I(X^n;Y^n) \text{(data processing ineq. } W \to X^n(W) \to Y^n\text{)} \\
&\leq \sum_{i=1}^{n} I(X_i;Y_i) \quad \text{(See next page.)} \\
&\leq nC \quad \text{(definition of channel capacity)}
\end{aligned}$$

- That is, no error $\to R \leq C$.

# No error $\rightarrow R \leq C$

**Lemma 1** *Let $Y^n$ be the result of passing $X^n$ through a discrete memoryless channel of capacity $C$. Then for all $p(x^n)$,*
$I(X^n;Y^n) \leq nC.$

**Proof.**

$$\begin{aligned}
I(X^n;Y^n) &= H(Y^n) - H(Y^n|X^n) \\
&= H(Y^n) - \sum_{i=1}^{n} H(Y_i|Y_1,\ldots,Y_{i-1},X^n) \\
&= H(Y^n) - \sum_{i=1}^{n} H(Y_i|X_i) \quad \text{(definition of DMC)} \\
&\leq \sum_{i=1}^{n} H(Y_i) - \sum_{i=1}^{n} H(Y_i|X_i) = \sum_{i=1}^{n} I(Y_i;X_i) \leq nC
\end{aligned}$$

# 7.9 Fano's Inequality and the Converse to the Coding Theorem

# Fano's Inequality

**Theorem 6 (Fano's inequality)** *Let $X$ and $W$ have the same sample spaces $\mathcal{X} = \{1, 2, \ldots, M\}$ and have the joint p.m.f. $p(x, w)$. Let*

$$P_e = \Pr[X \neq W] = \sum_{x \in \mathcal{X}} \sum_{\substack{w \in \mathcal{X}, \\ w \neq x}} p(x, w).$$

*Then*

$$P_e \log(M - 1) + H(P_e) \geq H(X|W)$$

*where*

$$H(P_e) = -P_e \log P_e - (1 - P_e) \log(1 - P_e).$$

# Fano's Inequality

**Proof.** We will prove that $H(X|W) - H(P_e) - P_e \log(M-1) \le 0$.

$$
\begin{aligned}
H(X|W) &= \sum_x \sum_w p(x,w) \log \frac{1}{p(x|w)} \\
&= \sum_x \sum_{w \neq x} p(x,w) \log \frac{1}{p(x|w)} \\
&\quad + \sum_x \sum_{w = x} p(x,w) \log \frac{1}{p(x|w)} \\
-P_e \log(M-1) &= \sum_x \sum_{w \neq x} p(x,w) \log \frac{1}{M-1} \\
-H(P_e) &= P_e \log P_e + (1-P_e) \log(1-P_e) \\
&= \sum_x \sum_{w \neq x} p(x,w) \log P_e \\
&\quad + \sum_x \sum_{w = x} p(x,w) \log(1-P_e)
\end{aligned}
$$

Add the above three terms together.

# Fano's Inequality

## Proof (cont.)

$$
\begin{aligned}
& H(X|W) - P_e \log(M-1) - H(P_e) \\
&= \sum_x \sum_{w \neq x} p(x,w) \log \frac{P_e}{(M-1)p(x|w)} + \sum_x \sum_{w=x} p(x,w) \log \frac{1-P_e}{p(x|w)} \\
&\leq \log \left( \sum_x \sum_{w \neq x} p(x,w) \frac{P_e}{(M-1)p(x|w)} + \sum_x \sum_{w=x} p(x,w) \frac{1-P_e}{p(x|w)} \right) \\
&= \log \left( \frac{P_e}{M-1} \sum_x \sum_{w \neq x} p(w) + (1-P_e) \sum_x \sum_{w=x} p(w) \right) \\
&= \log[P_e + (1-P_e)] = 0 \quad \square
\end{aligned}
$$

# Fano's Inequality

**Corollary 1** *1.* $P_e \log M + H(P_e) \geq H(X|W)$, $P_e = \Pr[X \neq W]$

*2.* $1 + P_e \log M \geq H(X|W)$, $P_e = \Pr[X \neq W]$

*3. If* $X \to Y \to \hat{X}$ *and* $P_e = \Pr[X \neq \hat{X}]$, *then*

$$H(P_e) + P_e \log M \geq H(X|\hat{X}) \geq H(X|Y)$$

**Remark.**

1. $H(X|W) \leq P_e \log(M-1) + H(P_e) \leq P_e \log M + H(P_e)$.
2. $H(X|W) \leq P_e \log(M-1) + H(P_e) \leq P_e \log M + 1$.
3. The second ineq. can be obtained by data processing ineq.

# Data Processing Inequality

**Lemma 2 (Data processing inequality)** *If* $X \to Y \to Z$, *then* $I(X;Z) \le I(X;Y)$

**Proof.**

$$
\begin{aligned}
& I(X;Z) - I(X;Y) \\
=& H(X) - H(X|Z) - [H(X) - H(X|Y)] = H(X|Y) - H(X|Z) \\
=& \sum_x \sum_y p(x,y) \log \frac{1}{p(x|y)} - \sum_x \sum_z p(x,z) \log \frac{1}{p(x|z)} \\
=& \sum_x \sum_y \sum_z p(x,y,z) \log \frac{1}{p(x|y)} - \sum_x \sum_y \sum_z p(x,y,z) \log \frac{1}{p(x|z)} \\
\le& \log \left( \sum_x \sum_y \sum_z p(x,y,z) \frac{p(x|z)}{p(x|y)} \right) \quad \text{(by convexity of logarithm)}
\end{aligned}
$$

# Data Processing Inequality

**Proof (cont.)** Since $X \to Y \to Z$, we have

$$p(x, y, z) = p(x, y)p(z|x, y) = p(x, y)p(z|y) = \frac{p(x, y)p(y, z)}{p(y)}$$

and

$$p(x, y, z)\frac{p(x|z)}{p(x|y)} = \frac{p(x, y)p(y, z)}{p(y)} \times \frac{p(x, z)p(y)}{p(z)p(x, y)} = \frac{p(x, z)p(y, z)}{p(z)}$$

Therefore,

$$\sum_x \sum_y \sum_z p(x, y, z)\frac{p(x|z)}{p(x|y)} = \sum_x \sum_y \sum_z \frac{p(x, z)p(y, z)}{p(z)}$$

$$= \sum_x \sum_z \frac{p(x, z)}{p(z)} \sum_y p(y, z) = \sum_x \sum_z p(x, z) = 1 \quad \square$$

# Data Processing Inequality (Summary)

**Lemma 3** *1. If $X \to Y \to Z$, then*

$$I(X;Z) \leq \begin{cases} I(X;Y) \\ I(Y;Z) \end{cases},$$

$$H(X|Y) \leq H(X|Z)$$

*2. If $X \to Y \to Z \to W$, then*

$$I(X;Z) + I(Y;W) \leq I(X;W) + I(Y;Z),$$

$$I(X;W) \leq I(Y;Z)$$

# Achievable $\to R \le C$

**Theorem 7 (Converse to Channel coding theorem)** *Any sequence of $(2^{nR}, n)$ codes with $\lambda^{(n)} \to 0$ must have $R \le C$.*

**Proof.**

- For a fixed encoding rule $X^n(W)$ and a fixed decoding rule $\hat{W} = g(Y^n)$, we have $W \to X^n(W) \to Y^n \to \hat{W}$.
- For each $n$, let $W$ be drawn according to a uniform distribution= over $\{1, 2, \ldots, 2^{nR}\}$.
- Since $W$ has a uniform distribution,

$$\Pr[W \neq \hat{W}] = P_e^{(n)} = \frac{1}{2^{nR}} \sum_{i=1}^{2^{nR}} \lambda_i.$$

# Achievable $\rightarrow R \leq C$

**Proof (cont.)**

$$
\begin{aligned}
nR &= H(W) \quad (W \text{ is uniform distribution}) \\
&= H(W|\hat{W}) + I(W;\hat{W}) \\
&\leq 1 + P_e^{(n)} nR + I(W;\hat{W}) \quad \text{(Fano's ineq.)} \\
&\leq 1 + P_e^{(n)} nR + I(X^n;Y^n) \quad \text{(data processing ineq.)} \\
&\leq 1 + P_e^{(n)} nR + nC \quad \text{(lemma 7.9.2)}
\end{aligned}
$$

$$
\Rightarrow P_e^{(n)} \geq 1 - \frac{C}{R} - \frac{1}{nR}
$$

That is, if $R > C$, the probability of error is large than a positive value for sufficiently large $n$. The error probability can't achieve arbitrary small. $\square$