Chapter 3

General Random Variables

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Chapter Contents

3.1 Continuous Random Variables and PDFs

Concepts

- Let X be the arrival time of a bus. X is a continuous random variable taking value between p and q.
- The sample space $\Omega = [p, q]$. For any point $c \in \Omega$, $\mathrm{P}(X = a) = 0$.
- We can find $\mathrm{P}(x \leq a)$, $\mathrm{P}(a < X \leq b)$, or $\mathrm{P}(a < X \leq a + \delta)$. In general, we can find $\mathrm{P}(X \in B)$ for $B \subset \Omega$.
- Let $F_X(x) = \mathrm{P}(X \leq x)$. We know that

$$F_X(-\infty) = \mathrm{P}(X \leq -\infty) = 0, \quad F_X(\infty) = \mathrm{P}(X \leq \infty) = 1$$

Concepts

- Let

$$f_X(x) \triangleq \lim_{\delta \to 0} \frac{F_X(x+\delta) - F_X(x)}{\delta} = \frac{dF_X(x)}{dx}$$

- We have

$$\mathrm{P}(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

- Therefore, $\mathrm{P}(X \in B)$ can be evaluated in terms of integral of $f_X(x)$. For example,

$$\mathrm{P}(a < X \leq b) = \int_a^b f_X(x)dx$$

PDF

- $f_X(x)$ is called the probability density function (PDF) of continuous random variable X.
- $F_X(x) = \mathrm{P}(X \le x)$ is called the cumulative distribution function (CDF).
  - $f_X(x) \ge 0$
  - $\int_{-\infty}^{\infty} f_X(x)dx = \mathrm{P}(-\infty < X \le \infty) = 1$
  - $\mathrm{P}(x < X < x + \delta) \approx f_X(x) \cdot \delta$ if δ is small.

Example 3.1. Uniform distribution.

$$f_X(x) = \begin{cases} c, & a \le x \le b \\ 0, & \text{otherwise.} \end{cases}$$

Find c.

Example 3.3. Uniform distribution.

$$f_X(x) = \frac{c}{\sqrt{x}}, \quad 0 < x \leq 1$$

Find c.

- A PDF can take arbitrarily large values.

Expectation

- The mean or expectation of a continuous random variable X is defined by $E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$.
- The kth moment is $E[X^k] = \int_{-\infty}^{\infty} x^k f_X(x) dx$.
- The variance of X is

$$\mathrm{Var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

- The mean of new RV $Y = g(X)$ is

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

Expectation

- The expectation is well-defined if

$$E[|X|] = \int_{-\infty}^{\infty} |x| f_X(x) dx < \infty.$$

- A not-well-defined random variable: Cauchy RV. Its PDF is

$$f_X(x) = \frac{c}{1+x^2}, \quad -\infty < x < \infty.$$

(Please find c). $E[X]$ is not well-defined.

Example 3.4. Uniformly RV

$$f_X(x) = \frac{1}{b-a}, \quad a \le x \le b.$$

Find $E[X]$ and $\mathrm{Var}(X)$.

Exponential RV

$$f_X(x) = ke^{-\lambda x}, \quad 0 \le x < \infty.$$

Find k, $E[X]$ and $\text{Var}(X)$.

3.2 Cumulative Distribution Functions

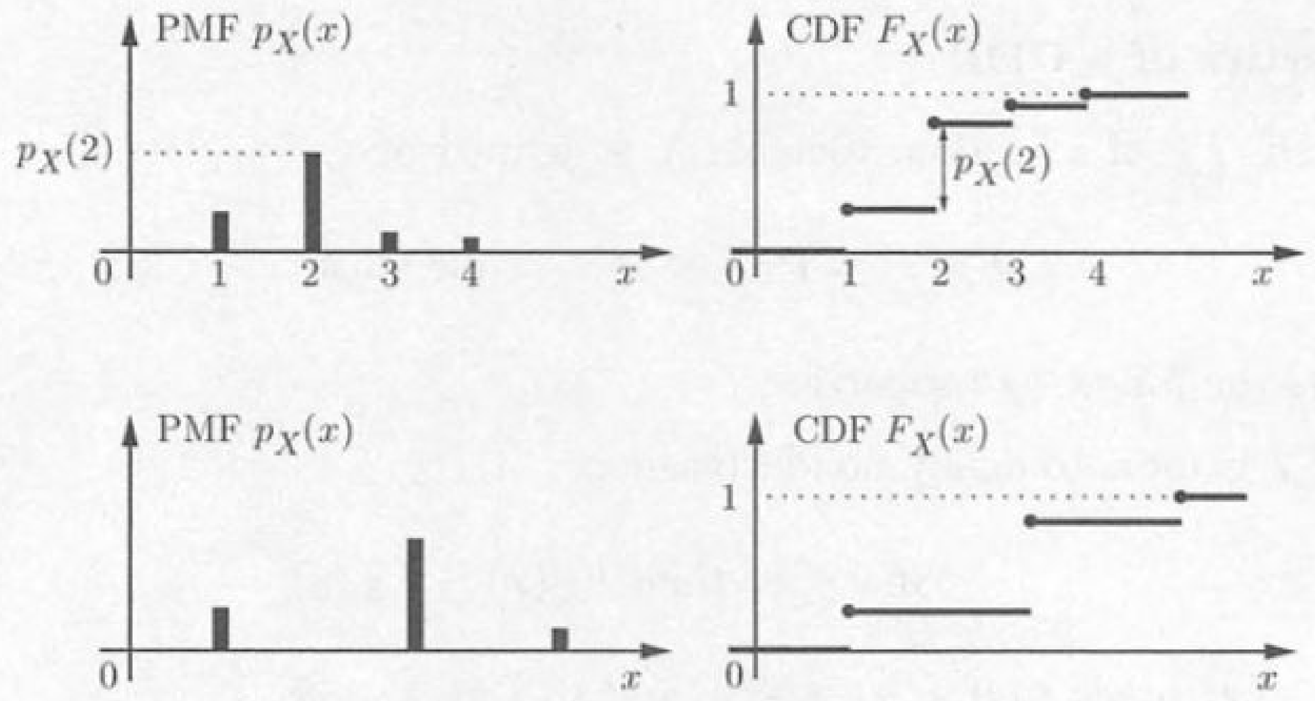
Cumulative Distribution Functions

- The cumulative distribution function (CDF) of a rv X, denoted by $F_X(x)$, is defined by

$$F_X(x) \triangleq \mathrm{P}(X \leq x) = \begin{cases} \sum_{k \leq x} p_X(k), & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{x} f_X(x) dx, & \text{if } X \text{ is continuous.} \end{cases}$$

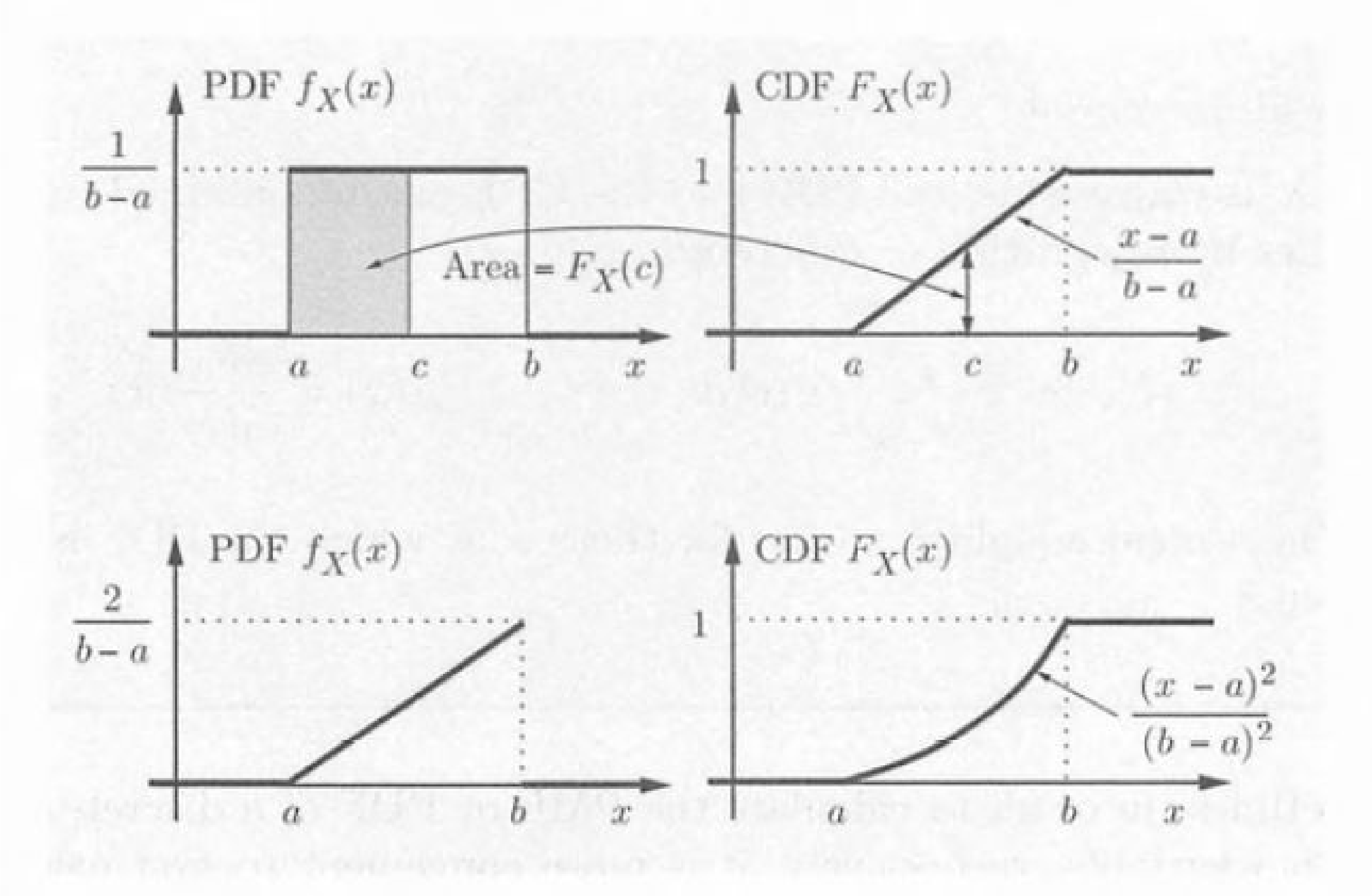
- CDF is exactly probability. PDF is NOT probability.
- Note the " $\leq$ " in the definition.
- "Any random variable associated with a given probability model has a CDF, regardless of whether it is discrete or continuous."

Fig 3.6



CDFs of some discrete random variables

Fig 3.7



CDFs of some continuous random variables

Properties of a CDF

- Definition: $F_X(x) \triangleq \mathrm{P}(X \leq x)$
- Monotonically nondecreasing: If $x \leq y$, then $F_X(x) \leq F_X(y)$.
- $F_X(-\infty) = 0$, $F_X(+\infty) = 1$
- If X is discrete, $F_X(x)$ is piecewise constant. If X is continuous, $F_X(x)$ is continuous.
- If X is discrete,

$$p_X(k) = F_X(k) - F_X(k-1)$$

- If X is continuous,

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Example 3.6

- Let X_1, X_2 and X_3 be 3 independent discrete random variables with identical PMFs.

$$X = \max\{X_1, X_2, X_3\}$$

Find PMF of X.

- Let X_1, X_2 and X_3 be 3 independent continuous random variables with identical PDFs.

$$X = \max\{X_1, X_2, X_3\}$$

Find PDF of X.

3.3 Normal Random Variables

Definition

A continuous random variable X is said to be **normal** or **Gaussian** if it has a PDF of the form

$$f_X(x) = ce^{-(x-\mu)^2/(2\sigma^2)}, -\infty < x < \infty.$$

- $c = \frac{1}{\sqrt{2\pi}\sigma}$
- $E[X] = \mu$
- $\mathrm{Var}(X) = \sigma^2$

Standard Normal

- A standard normal rv Y is the normal rv with $\mu = 0$ and $\sigma = 1$. Its CDF is denoted by $\Phi(y)$

$$\Phi(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

- $\Phi(-y) = 1 - \Phi(y)$ because The PDF of standard normal is even.
- For normal rv X with mean μ and variance σ^2, we know that $X = \sigma Y + \mu$. Thus,

$$\mathrm{P}(X \le x) = \mathrm{P}(\sigma Y + \mu \le x) = \mathrm{P}\left(Y \le \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Example 3.7

The annual snowfall at a particular geographic location is modeled as a normal random variable with a mean of $\mu = 60$ inches and a standard deviation of $\sigma = 20$. What is the probability that this year's snowfall will be at least 80 inches?

$Y = 20X + 60$
$P(Y \geq 80) = ?$

Example 3.8

A binary message is transmitted as a signal s, which is either $-l$ or $+1$. The communication channel corrupts the transmission with additive normal noise N with mean $\mu = 0$ and variance σ^2. The receiver concludes that the signal -1 (or $+1$) was transmitted if the value received is < 0 (or ≥ 0, respectively). What is the probability of error?

3.4 Joint PDFs Of Multiple Random Variables

Definitions

- X and Y are two continuous random variables. Their joint CDF is

$$F_{X,Y}(x,y) \triangleq \mathrm{P}(X \leq x, Y \leq Y)$$

- Joint PDF is

$$f_{X,Y}(x,y) \triangleq \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

Properties

1. $\mathrm{P}((X,Y)\in B)=\iint_{(x,y)\in B} f_{X,Y}(x,y)dxdy$

2. $\mathrm{P}(a<X\le b, c<Y\le d)=\int_{x=a}^{b}\int_{y=c}^{d} f_{X,Y}(x,y)dydx$

3. $F_{X,Y}(x,y)=\int_{s=-\infty}^{x}\int_{t=-\infty}^{y} f_{X,Y}(s,t)dtds$

4. $\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f_{X,Y}(x,y)dydx=1$

5. $\mathrm{P}(x<X\le x+\delta, y<Y\le y+delta)\approx f_{X,Y}(x,y)\delta^2$

Marginal PDF

$$F_X(x) = P(X \le x) = P(X \le x, -\infty < Y < \infty)$$
$$= \int_{s=-\infty}^{x} \int_{y=-\infty}^{\infty} f_{X,Y}(s,y)\,dy\,ds$$
$$\Rightarrow f_X(x) = \frac{dF_X(x)}{dx} = \int_{y=-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

Similarly,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

Example 3.9 Jointly Uniform PDF

$$f_{X,Y}(x,y) = c,$$

- If $a < x < b$ and $c < y < d$, find c.
- If $|x| + |y| \leq r$, find c.
- If $\sqrt{x^2 + y^2} \leq r$, find c.

Expectation

$$E[g(X,Y)] = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dy\,dx$$

- $E[aX + bY + c] = aE[X] + bE[Y] + c$
- You can easily extend the results of two random variables to more joint random variables.

3.5 Conditioning

Conditioning an RV on an Event

- Conditioning CDF and PDF

$$F_{X|A}(x) = \mathrm{P}(X \le x|A) = \frac{\mathrm{P}(\{X \le x\} \cap A)}{\mathrm{P}(A)}$$

$$f_{X|A}(x) = \frac{d}{dx} F_{X|A}(x)$$

- Special case: $A = \{X \in B\}$

$$F_{X|X\in B}(x) = \mathrm{P}(X \le x|X \in B) = \frac{\mathrm{P}(\{X \le x\} \cap \{X \in B\})}{\mathrm{P}(X \in B)}$$

$$= \frac{\int_{t \le x, t \in B} f_X(t) dt}{\mathrm{P}(X \in B)}$$

$$f_{X|X\in B}(x) = \frac{d}{dx} F_{X|X\in B}(x) = \frac{f_X(x)}{\mathrm{P}(X \in B)}, \quad x \in B.$$

Example 3.13. The Exponential RV

The time Tuntil a new light bulb burns out is an exponential rv with parameter λ. You turn the light on, leaves the room, and when you returns, t time units later, finds that the light bulb is still on, which corresponds to the event $A = \{T > t\}$. Let X be the additional time until the light bulb burns out. What is the conditional CDF of X, given the event A?

Hint. $\mathrm{P}(X > x|A) = \mathrm{P}(T > T + x|T > t) =?$

Example 3.14.

The metro train arrives at the station near your home every quarter hour starting at 6:00 a.m. You walk into the station every morning between 7:10 and 7:30 a.m., and your arrival time is a uniform random variable over this interval. What is the PDF of the time you have to wait for the first train to arrive?

Hint. Let X be the time of your arrival. X is a uniform random variable over the interval from 7:10 to 7:30. Let Y be the waiting time. Let A and B be the events

$$A = \{7:10 \le X \le 7:15\} = \{\text{you board the 7:15 train}\},$$

$$B = \{7:15 < X \le 7:30\} = \{\text{you board the 7:30 train}\}.$$

$$f_Y(y) = \mathrm{P}(A) f_{Y|A}(y) + \mathrm{P}(B) f_{Y|B}(y)$$

Conditioning one RV on Another

- Let X and Y be continuous random variables with joint PDF $f_{X,Y}(x,y)$. The conditional PDF of X given that $Y = y$, is defined by

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Example 3.15.

$$f_{X,Y}(x,y) = c, \quad x^2 + y^2 \leq r^2$$

Find c, $f_Y(y)$ and $f_{X|Y}(x|y)$.

Conditional Expectations

- Definition.

$$E[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x)dx$$

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx$$

- Total expectation. Let $A_1, A_2, \ldots, A_n$ form a partition of the sample space.

$$E[X] = \sum_i \mathrm{P}(A_i)E[X|A_i]$$

$$E[X] = \int_{-\infty}^{\infty} f_Y(y)E[X|Y=y]dy = E_Y[E[X|Y=y]]$$

Independence

- Two continuous random variables X and Y are independent if

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

- Three continuous random variables X, Y and Y are independent if

$$f_{X,Y,Z}(x, y, z) = f_X(x) f_Y(y) f_Z(z)$$

- If X and Y are independent, we have

$$E[XY] = E[X]E[Y]$$