Chapter 4

Further Topics on Random Variables

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4.1 Derived Distributions

Concepts

- Let X be an RV with pdf $f_X(x)$ and $Y = g(X)$.

$$F_Y(y) = \mathrm{P}(Y \le y) = \mathrm{P}(g(X) \le y) = \int_{\{x|g(x)\le y\}} f_X(x)dx.$$

$$f_Y(y) = \frac{dF_Y(y)}{dy}$$

Example 4.1.

Let X be uniform rv on $[0,1]$ and $Y = \sqrt(X)$. Find CDF and PDF of Y.

Example 4.3.

Let X be an rv with PDF $f_X(x)$. and $Y = X^2$. Find PDF of Y in terms of PDF of X.

Example.

Let X be an rv with PDF $f_X(x)$. and $Y = aX + b$. Find PDF of Y in terms of PDF of X.

Hint. Note the sign of a.

Example 4.5

Let X be a normal rv with PDF mean μ and variance σ^2.
Let $Y = aX + b$. Find PDF of Y.

Example 4.7

Let X and Y be two independent uniform rvs on $[0, 1]$. Let $Z = \max(X, Y)$. Find PDF of Y.

Example 4.8

Let X and Y be two independent uniform rvs on $[0, 1]$. Let $Z = Y/X$. Find PDF of Y.

Example 4.9

Let X and Y be two independent exponential rvs with parameter λ. Let $Z = X - Y$. Find PDF of Y.

Sums of Independent RVs

- Let X and Y be two independent discrete rvs with PMFS $p_X(x)$ and $p_Y(y)$. Let $Z = X + Y$.

$$p_Z(z) = \mathrm{P}(X + Y = z) = \sum_{x=k, y=z-k} p_X(k) p_Y(z-k)$$

- Let X and Y be two independent continuous rvs with PDFS $f_X(x)$ and $f_Y(y)$. Let $Z = X + Y$.

$$\mathrm{P}(Z \le z | X = x) = \mathrm{P}(X + Y \le z | X = x) = \mathrm{P}(x + Y \le z) = \mathrm{P}(Y \le z - x)$$

$$\Rightarrow f_{Z|X}(z|x) = f_Y(z-x)$$

$$\text{since } f_{Z,X}(z,x) = f_X(x) f_{Z|X}(z|x) = f_X(x) f_Y(z-x)$$

$$\Rightarrow f_Z(z) = \int_{-\infty}^{\infty} f_{Z,X}(z,x)dx = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx$$

- "convolution" or convolutional sum/integral.

Example 4.10

Let X and Y be independent rvs uniformly distributed on $[0, 1]$ and let $W = X + Y$. Find PDF of W.

4.2 Covariance and Correlation

Definition

- The **covariance** of two rvs X and Y is defined by

$$\text{cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$$

- **uncorrelated:** $\text{cov}(X,Y) = 0$
- "Independent" implies "uncorrelated".
- The **correlation coefficient** of two rvs X and Y is defined by

$$\rho = \frac{\text{cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

- $-1 \le \rho \le 1$
- $\rho^2 = 1$ if and only if $X - E[X] = c(Y - E[Y])$

Properties

- $\text{cov}(X, Y) = E[XY] - E[X]E[Y]$
- $\text{cov}(X, X) = \text{var}(X)$
- $\text{cov}(X, aY + b) = a \times \text{cov}(X, Y)$
- $\text{cov}(X, Y + Z) = \text{cov}(X, Y) + \text{cov}(X, Z)$

Example 4.13

Let X and Y be random variables with joint PMF of

$$P_{X,Y}(0,1) = P_{X,Y}(1,0) = P_{X,Y}(0,-1) = P_{X,Y}(-1,0) = \frac{1}{4}$$

- $P_X(-1) = 1/4$, $P_X(0) = 1/2$, $P_X(1) = 1/4$
- $P_Y(-1) = 1/4$, $P_Y(0) = 1/2$, $P_Y(1) = 1/4$
- $E[X] = E[Y] = 0$, $E[XY] = 0$
- $\text{cov}(X,Y)] = E[XY] - E[X]E[Y] = 0$
- X and Y are uncorrelated, not independent.

Example 4.14

Consider n independent tosses of a coin with probability of a head equal to p. Let X and Y be the numbers of heads and of tails, respectively. Calculate the correlation coefficient $\rho(X,Y)$ of X and Y.

Since $X+Y=n$, we have $E[X]+E[Y]=n$, and $X-E[X]=-(Y-E[Y])$. Therefore,

$$\begin{aligned}\text{cov}(X,Y) &= E[(X-E[X])(Y-E[Y])] \\ &= -E[(X-E[X])^2] = -\text{var}(X) \\ \Rightarrow \rho(X,Y) &= \frac{\text{cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{-\text{var}(X)}{\sqrt{\text{Var}(X)\text{Var}(X)}} = -1\end{aligned}$$

4.3 Conditional Expectation As A Random Variable

Law of iterated expectations

$$
\begin{aligned}
E[X] &= \int_x x f_X(x) dx = \int_x \int_y x f_{X,Y}(x,y) dy dx \\
&= \int_y \int_x x f_{X|Y}(x|y) dx f_Y(y) dy \\
&= \int_y E[X|Y=y] f_Y(y) dy = E[E[X|Y]]
\end{aligned}
$$

- $E[X|Y=y]$: a function of y, a deterministic function of y
- $E[X|Y]$: a function of Y, a derived random variable from Y

Example

Let X and Y be two continuous random variables uniformly distributed on the region specified by $x \geq 0, y \geq 0, x + y \leq 1$.

- Find their joint pdf $f_{X,Y}(x, y)$.
- Find $f_{X|Y}(x|y)$ and $E[X|Y = y]$.
- Find $E[X]$

Example 4.17

We start with a stick of length ℓ. We break it at a point which is chosen randomly and uniformly over its length, and keep the piece that contains the left end of the stick. We then repeat the same process on the piece that we were left with. What is the expected length of the piece that we are left with after breaking twice?

Let Y be the length of the piece after we break for the first time. Let X be the length after we break for the second time. We have

$$E[X|Y] = Y/2, \quad E[Y] = \ell/2$$

therefore, $E[X] = E[E[X|Y]] = E[Y/2] = E[Y]/2 = \ell/4$

Law of iterated Variance

$$X - E[X] = (X - E[X|Y]) + (E[X|Y] - E[X])$$

$$\begin{aligned}
\mathrm{Var}(X) &= E[(X - E[X])^2 = E[((X - E[X|Y]) + (E[X|Y] - E[X]))^2] \\
&= E[(X - E[X|Y])^2) + E[(E[X|Y] - E[X])^2] \\
&+ 2E[(X - E[X|Y])(E[X|Y] - E[X])] \\
&= E[E[(X - E[X|Y])^2)|Y] + E[(E[X|Y] - E[E[X|Y]])^2] \\
&= E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y])
\end{aligned}$$

where we use the fact of $E[E[X|Y]h(Y)] = E[E[Xh(Y)|Y]] = E[Xh(Y)]$ to deduce

$$\begin{aligned}
&E[(X - E[X|Y])(E[X|Y] - E[X])] = E[XE[X|Y] - XE[X] - E[X|Y]^2 + E[X]E[X|Y]] \\
=&E[XE[X|Y]] - E[X]^2 - E[E[X|Y]^2] + E[E[X]E[X|Y]] \\
=&E[XE[X|Y]] - E[X]^2 - E[E[XE[X|Y]|Y]] + E[X]^2 \\
=&E[XE[X|Y]] - E[XE[X|Y]] = 0
\end{aligned}$$

4.4 Transforms

Moment Generating Function

- Transform is another representation of probability law.
- Transform is a mathematical tool for facilitating some manipulations.
- The moment generating function (MGF) is one of many transforms, defined by

$$M_X(s) = E[e^{sX}] = \begin{cases} \sum_k e^{sk} p_X(k), & X \text{ is discrete;} \\ \int_{-\infty}^{\infty} e^{sx} f_X(x) dx, & X \text{ is continuous.} \end{cases}$$

Example 4.22

Let $p_X(2) = 1/2$, $p_X(3) = 1/6$, $p_X(5) = 1/3$. Find MGF of X.

Example 4.23

Find MGF of Poisson random variable with parameter λ.

Example 4.24

Find MGF of exponential random variable with parameter λ.

Example 4.25

Let $Y = aX + b$. Express MGF of Y in terms of MGF of X.

Example 4.26

Find MGF of normal random variable with mean μ and variance σ^2.

Usage of MGF

$$M_X(s) = E[e^{sX}] = E\left[1 + sX + \frac{s^2X^2}{2!} + \frac{s^3X^3}{3!}\right]$$

$$\Rightarrow E[X] = \left.\frac{dM_X(s)}{ds}\right|_{s=0}$$

$$\Rightarrow E[X^2] = \left.\frac{d^2M_X(s)}{ds^2}\right|_{s=0}$$

$$\Rightarrow E[X^n] = \left.\frac{d^nM_X(s)}{ds^n}\right|_{s=0}$$

Example 4.27

Let $p_X(2) = 1/2$, $p_X(3) = 1/6$, $p_X(5) = 1/3$. Find MGF of X and use MGF to evaluate $E[X]$ and $E[X^2]$.

Inversion Property

- If $M_X(s) = M_Y(s)$ for all S, then X and Y have the same probability law.
- Let X and Y are independent RVs. If $Z = X + Y$, then $M_Z(s) = M_X(s)M_Y(s)$.

Example 4.28

The MGF of an RV X is given by

$$M_X(s) = \frac{1}{4}e^{-s} + \frac{1}{2} + \frac{1}{8}e^{4s} + \frac{1}{8}e^{5s}.$$

Find its PMF.

Example 4.29

The MGF of an RV X is given by

$$M_X(s) = \frac{pe^s}{1-(1-p)e^s},$$

Find its PMF.

Sums of Independent RVs

Let X and Y be independent random variables.
$W = X + Y$. Then

$$M_W(s) = E[e^{sW}] = E[e^{s(X+Y)}] = E[e^{sX}]E[e^{sY}] = M_X(s)M_Y(s)$$

Example. Let $X_1, X_2, ...X_n$ be independent Bernoullui rvs with parameter p. $Y = X_1 + X_2 + \cdots + X_n$. Find MGF of Y.