CHAPTER 5
Integration

1. Antidifferentiation: The Indefinite Integral
2. Integration by Substitution
3. The Definite Integral and the Fundamental Theorem of Calculus
4. Applying Definite Integration: Area Between Curves and Average Value
5. Additional Application to Business and Economics
6. Additional Application to the Life and Social Sciences

Recall things learned before

- Basic concepts of functions (limit, continuity)
- Types of functions (polynomial, exponential, log)
- Tangent lines and secant lines for functions
- Derivative of the function
- Techniques of derivative
- Applications of derivative

5.1 Antidifferentiation （反微分，反導數）：The Indefinite Integral（不定積分）

Motivations for antidiffertiation

- Examples for antidiffertiation
  - How can a known rate of inflation be used to determine future prices?
  - What is the velocity of an object moving along a straight line with known acceleration?
  - How can knowing the rate at which a population is changing be used to predict future population levels?
- In all these situations, the derivative (rate of change) of a quantity is known and the quantity itself is required.
  - Here is the terminology we will use in connection with obtaining a function from its derivative.

Antidifferentiation

A function *F(x)* is said to be an ***antiderivative*** of *f(x)* if

$$F'(x) = f(x)$$

for every *x* in the domain of *f(x)*.即給定微分後的函數下，尋找未微分前的函數。

Note

The process of finding antiderivatives is called antidifferentiation or indefinite intergration.

Example 5.1.1

Verify that $F(x) = \frac{1}{3}x^3 + 5x + 2$ is an antiderivative of $f(x) = x^2 + 5$

Q: Only one antiderivative of f(x)? Find other antiderivatives of f(x).

Note: Try other examples

The general antiderivative of a function

- A function has more than one antiderivative.
- See Figure 5.1, p373.
- In general, if F is one antiderivative of f, then so is any function of the form $G(x)=F(x)+C$, for constant C. (why?)
- Conversely, it can be shown that if F and G are both antiderivatives of f, then $G(x)=F(x)+C$, for some constant C.

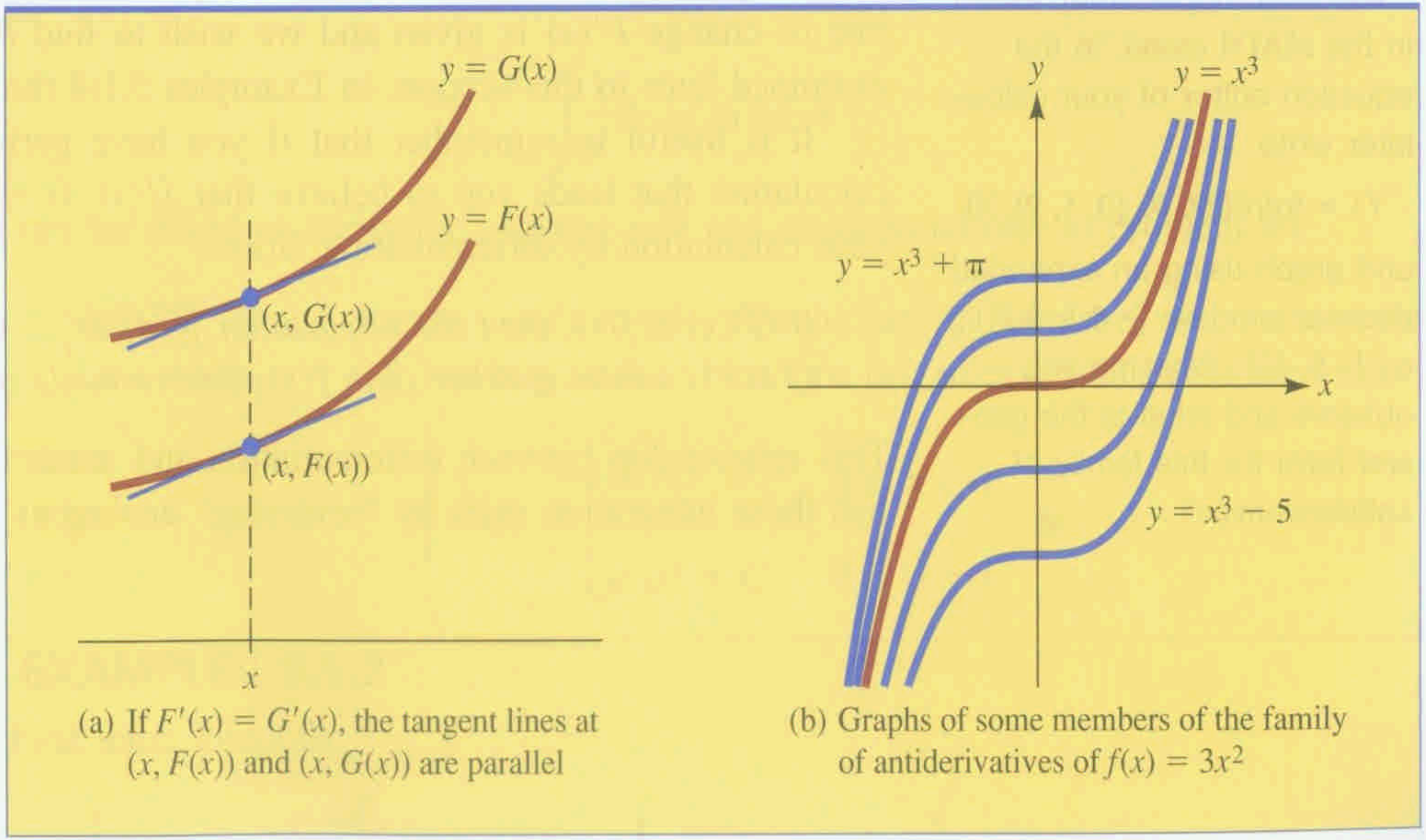


FIGURE 5.1 Graphs of antiderivatives of a function f form a family of parallel curves.

Fundamental property of antiderivatives

- If $F(x)$ is an antiderivative of the continuous function $f(x)$, then any other antiderivative of $f(x)$ has the form $G(x)=F(x)+C$ for some constant C.
- Fact 1: The slope of the tangent line to $y=F(x)$ at the point $(x, F(x))$ is the same as the slope of the tangent line to $y=G(x)$ at the point $(x, G(x))$.
- Fact 2: Since the slopes in Fact 1 are the same, the tangent lines at $(x, F(x))$ and $(x, G(x))$ are parallel (see Figure 5.1a).

Fundamental property of antiderivatives

- Fact 3: Since Fact 2 is true for all x, the entire curve $y=G(x)$ must be parallel to the curve $y=F(x)$, so that $y=G(x)=F(x)+C$.
- In general, the collection of graphs of all antiderivatives of a given function f is a family of parallel curves that are vertical translations of one another (see Figure 5.1b).

The indefinite intergral

- The family of all antiderivatives of f(x) is written

$$\int f(x)dx = F(x) + C \quad \text{for some constant } k$$

  and is called the indefinite integral of f(x). Here f(x) is called the integrand.

Note:

The integral is "indefinite" because it involves a constant C that take on any value, see other issues on p374.

Rules for Integrating Common Functions

- The **constant rule**: $\int k\,dx = kx + C$ for some constant k
- The **Power rule**: $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$ for all $n \neq -1$
- The **logarithmic rule**: $\int \dfrac{1}{x} dx = \ln|x| + C$ for all $x \neq 0$
- The **exponential rule**:

$$\int e^{kx} dx = \frac{1}{k} e^{kx} + C \text{ for constant } k \neq 0$$

Example 5.1.2

Find these integrals

a. $\int 3\,dx$ b. $\int x^{17}dx$ c. $\int \frac{1}{\sqrt{x}}dx$ d. $\int e^{-3x}dx$

Algebraic Rules for Indefinite Integration

- The **constant multiple rule**:

$$\int kf(k)dx = k\int f(x)dx \text{ for constant k}$$

- The **sum rule**:

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

- The **difference rule**:

$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

Example 5.1.3 (p377)

Find the following integrals

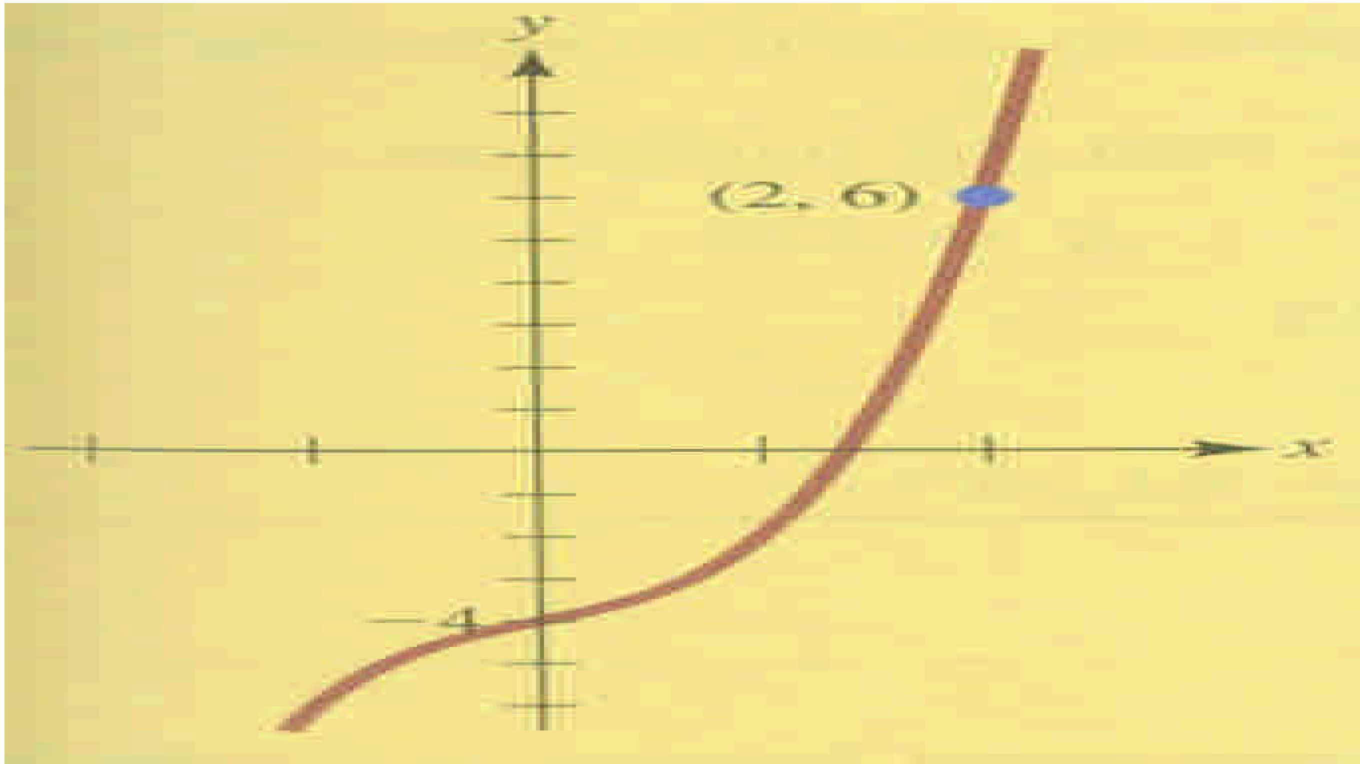
a. $\int (2x^5 + 8x^3 - 3x^2 + 5)dx$

b. $\int (\frac{x^3 + 2x - 7}{x})dx$

c. $\int (3e^{-5t} + \sqrt{t})dt$

Example 5.1.4

Find the function *f(x)* whose tangent has slope $3x^2+1$ for each value of x and whose graph passes through the point (2,6)



The graph of $y = x^3 + x - 4$.

Applied initial value problems
Example 5.1.5

A manufacturer has found that marginal cost is $3q^2 - 60q + 400$ dollars per unit when q units have been produced.

The total cost of producing the first 2 units is \$900.

Q: What is the total cost of producing the first 5 units?

Example 5.1.6

The population P(t) of a bacterial colony t hours after observation begins is found to be changing at the rate

$$\frac{dP}{dt} = 200e^{0.1t} + 150e^{-0.03t}$$

If the population was 200,000 bacteria when observations began, what will the population be 12 hours later?

Example 5.1.7

A retailer receives a shipment of 1,0000 kilograms of rice that will be used up over a 5-month period at the constant rate of 2,000 kilograms per month.

If storage costs are 1 cent per kilogram per month, how much will the retailer pay in storage costs over the next 5 months?

Example 5.1.8

A car is traveling along a straight, level road at 45 miles per hours (66 feet per second) when the driver is forced to apply the brakes to avoid an accident.

If the brakes supply a constant deceleration of 22 ft/sec^2 (feet per second, per second), how far does the car travel before coming to a complete stop?

5.2 Integration by substitution (替代法)

Process of integration by substitution

1. Choose a substitution $u = u(x)$ that "simplifies" the integrand $f(x)$.
2. Express the entire integral in terms of u and $du = u'(x)\, dx$. This means that all terms invloving x and dx must be transformed to terms involving u and du.
3. The given integral should have the form

$$\int \mathrm{f}(x)dx = \int g(u)du$$

Process of integration by substitution

4. Replace u by $u(x)$ in $G(u)$ to obtain an antiderivative $G(u(x))$ for $g(x)$, so that

$$\int f(x)dx = \int g(u(x))u'(x)dx = \int \left(\frac{d}{dx}\left[G(u(x)) \right] \right) dx = G(u(x)) + C$$

Note: The detail of the process could be found on p385-386

Examples 5.2.1-5.2.10

Find $\int \sqrt{2x+7}\,dx$ $\int 8x(4x^2-3)^5\,dx.$ $\int x^3 e^{x^4+2}\,dx$

$\int \frac{3x+6}{\sqrt{2x^2+8x+3}}\,dx$ $\int \frac{x}{x-1}\,dx$ $\int \frac{(\ln x)^2}{x}\,dx$

$\int e^{5x+2}\,dx$ $\int \frac{x^2+3x+5}{x+1}\,dx$ $\int \frac{1}{1+e^{-x}}\,dx$

$\int x^4 e^{x^4+2}\,dx$

Example 5.2.11

The price p (dollars) of each unit of a particular commodity is estimated to be changing at the rate

$$\frac{dp}{dx} = \frac{-135x}{\sqrt{9 + x^2}}$$

where x (hundred) units is the consumer demand (the number of units purchased at that price).

Suppose 400 units ($x=4$) are demanded when the price is \$30 per unit.

Example 5.2.11

a. Find the demand function $p(x)$.

b. At what price will 300 units be demanded? At what price will no units be demanded?

c. How many units are demanded at a price of \$20 per unit?

5.3 The Definite integral and the Fundamental Theorem of Calculus

Example for the motivation of definite integral

- Suppose a real estate agent wants to evaluate an unimproved parcel of land that is 100 feet wide and is bounded by streets on three sides and by a stream on the fourth side, see Figure 5.2.
- Q: How can the agent find the area of the parcel in order to decide the total value of the parcel?

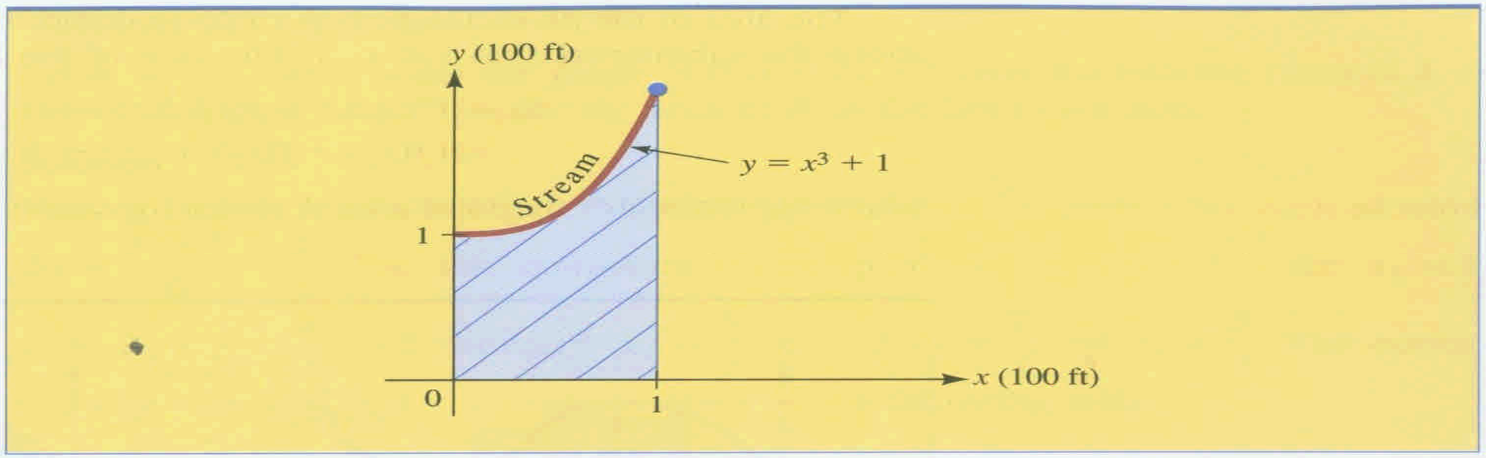


FIGURE 5.2 Determining land value by finding the area under a curve.

Area as the limit of a sum

- Consider the area of the region under the curve $y=f(x)$ over an interval $a \leqq x \leqq b$, where $f(x) \geqq 0$ and f is continuous, see Figure 5.3 (p398).
- To find this area, we will follow a useful general policy:
  - *When faced with something you don't know how to handle, try to relate it to something you do know how to handle.*
- We may not know the area under the given curve, but we do know how to find the area of a rectangle.

Process for finding the approximation of area under any curve by rectangles (p398).

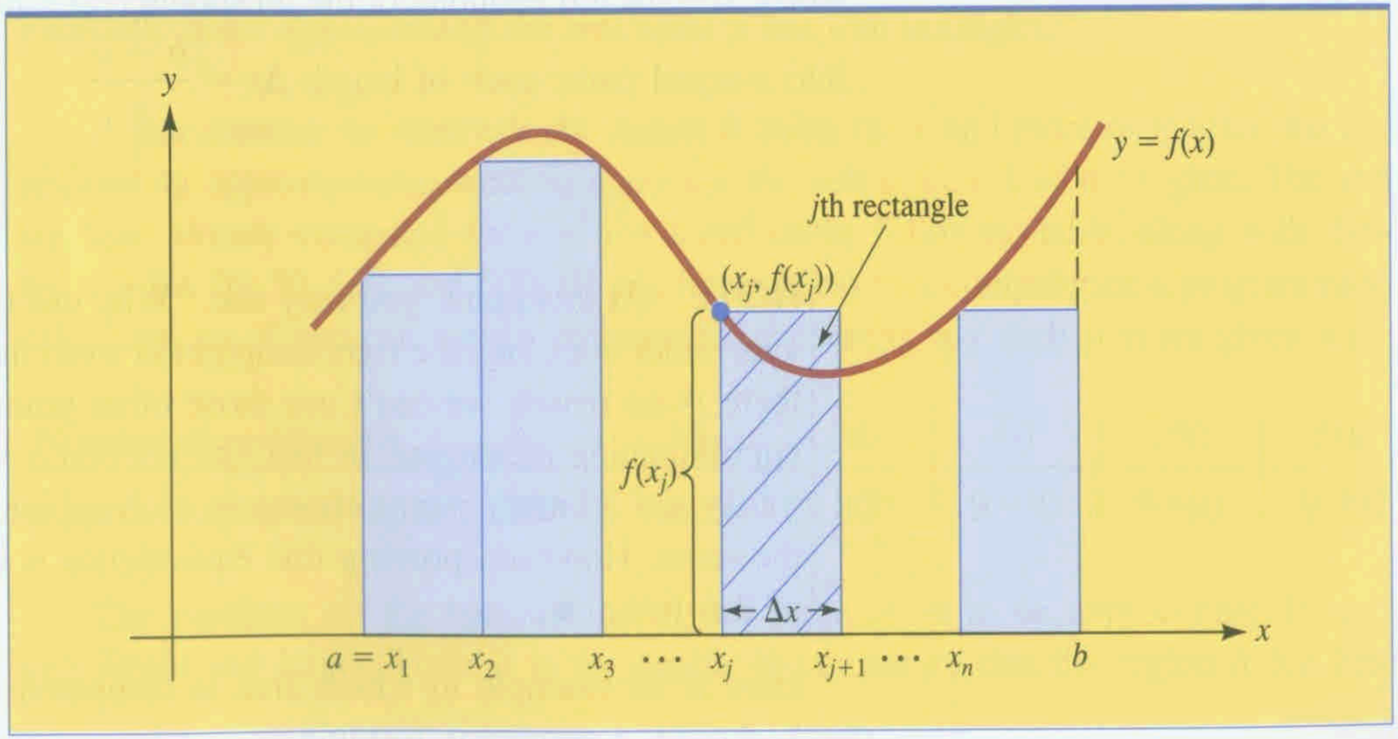


FIGURE 5.4 An approximation of area under a curve by rectangles.

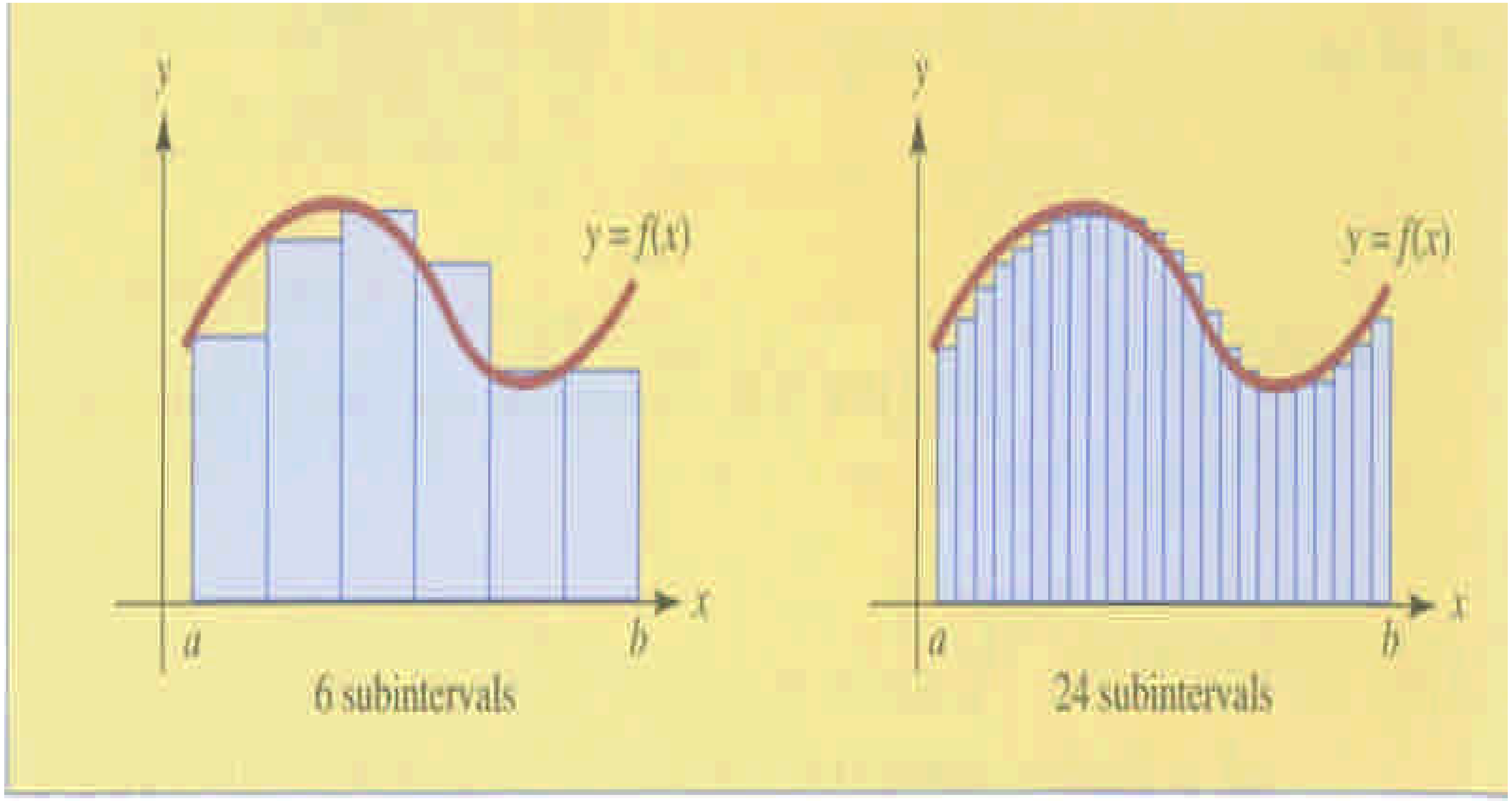


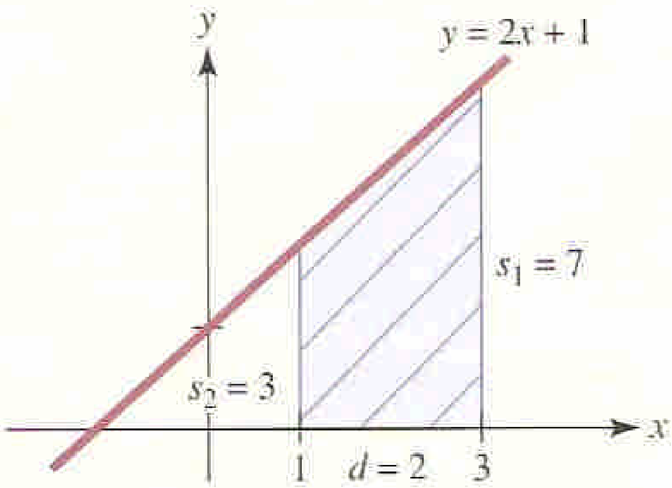
FIGURE 5.5 The approximation improves as the number of subintervals increases.

Q: Find any difference between these two figures?

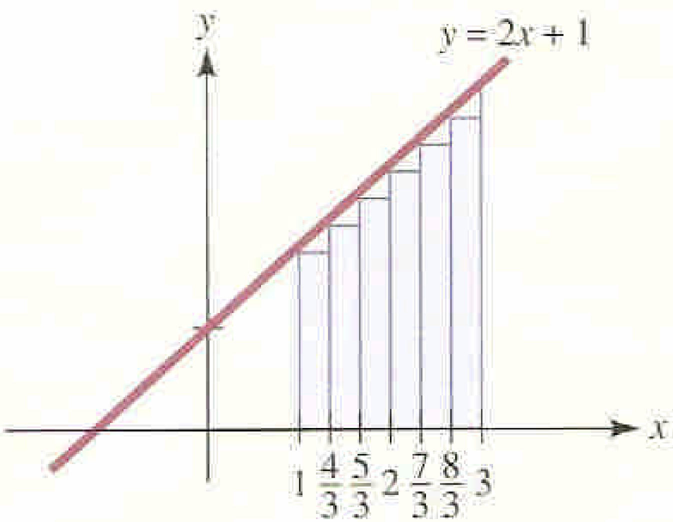
Formula for the Area under a curve (p399)

Example 5.3.1

Let R be the region under the graph of $f(x) = 2x + 1$ over the interval $1 \leqq x \leqq 3$, as shown if Figure 5.6a. Compute the area of R as the limit of a sum.



(a) The region R under the line $y = 2x + 1$ over $1 \leq x \leq 3$ (a trapezoid)



(b) The region R subdivided by six approximating rectangles

FIGURE 5.6 Approximating the area under a line with rectangles.

Importance:
Definition of the definite integral (p401)

The definite integral is the limit of the Riemann sum

Notes for definite integral

- $f(x)$ is continuous on the interval $a \leqq x \leqq b$
- Two steps for computing definite integral
  1. Compute Riemann sum
  2. Compute the limit of Riemann sum as $n \to \infty$
- The process of finding a definite integral is called definite integration.
- Riemann sum is a troublesome process!!!!

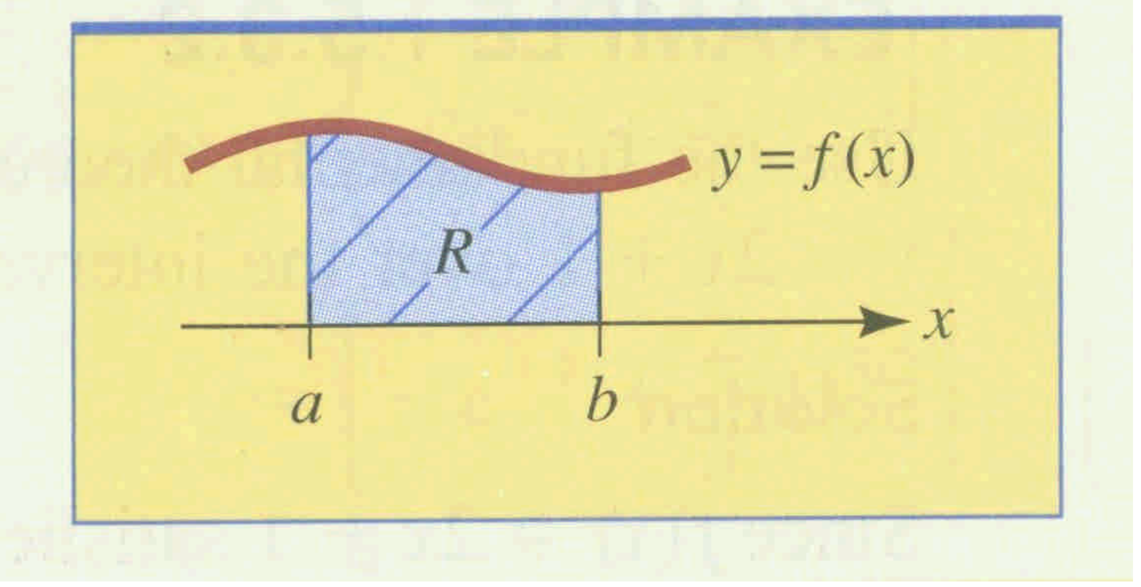
Notes for definite integral

- The symbol $\int_a^b f(x)dx$ used for the definite integral is essentially the same as the symbol $\int f(x)dx$ for the indefinite integral even though the definite integral is a specific number while the indefinite integral is a family of functions, the antiderivatives of f.
- In fact, these two apparently very different concepts are intimately (緊密的) related.

Area as a Definite Integral

If $f(x)$ is continuous and $f(x) \geq 0$ on the interval $a \leq x \leq b$, Then the region R under the curve $y = f(x)$ over the interval $a \leq x \leq b$ has area A given by the definite integral

$$A = \int_a^b f(x)dx$$



Fundamental Theorem of Calculus (FTC)

If the function *f(x)* is continuous on the interval $a \leq x \leq b$, then

$$\int_a^b f(x)dx = \mathrm{F}(x)\Big|_a^b = F(b) - F(a)$$

where *F(x)* is any antiderivative of *f(x)* on $a \leq x \leq b$.

Notes for FTC

- It is much easily to compute area under the curve with FTC, comparing to the Riemann sum, if the antiderivative of f(x) is available.
- Any antiderivative of f(x) could be applied to FTC and gains the same result.
- However, sometimes it is difficult to find the antiderivative of f(x)……..

Example 5.3.2

Use the fundamental theorem of calculus to find the area of the region under the line $y = 2x+1$ over the interval $1 \leq x \leq 3$.

Example 5.3.3

Find the area of the parcel of land described in the introduction to this section; that is, the area under the curve $y = x^3 + 1$ over the interval $0 \leq x \leq 1$, where x and y are in hundreds of feet.

If the land in the parcel is appraised at $12 per square foot, what is the total value of the parcel?

Example 5.3.4-5

Evaluate the definite integral

$$\int_0^1 \left(e^{-x} + \sqrt{x}\right)dx$$

$$\int_1^4 \left(\frac{1}{x} - x^2\right)dx$$

Integration Rules

Let f and g be any functions continuous on a≤x≤b. Then

- **Constant multiple rule**:

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx \text{ for constant k}$$

- **Sum rule**:

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

- **Difference rule**:

$$\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

Integration Rules

Let f and g be any functions continuous on a≤x≤b. Then

- $$\int_a^a f(x)dx = 0 \text{ and } \int_b^a f(x)dx = -\int_a^b f(x)dx$$

- **Subdivision rule**:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

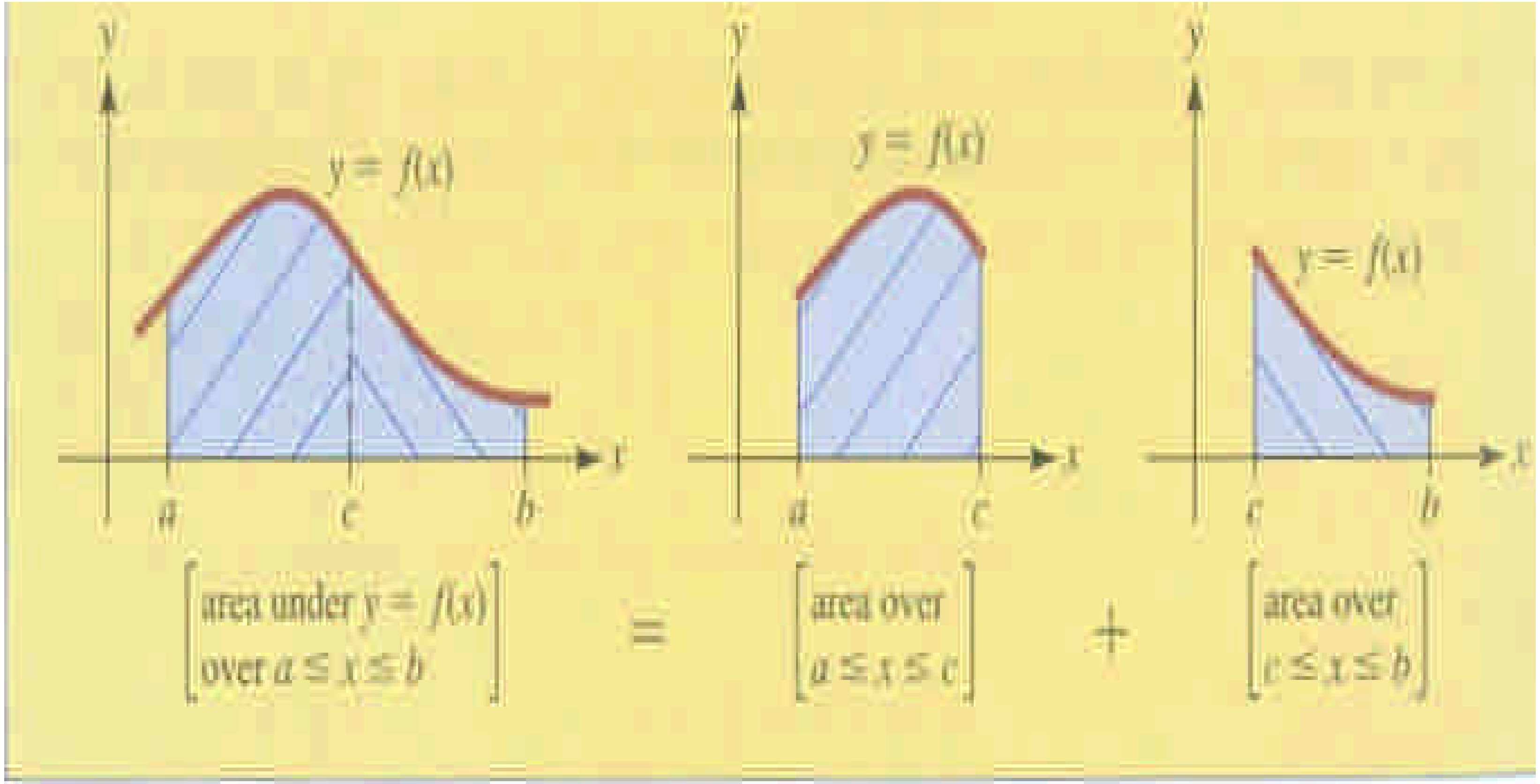


FIGURE 5.7 The subdivision rule for definite integrals (case where $f(x) \geq 0$).

Example 5.3.6-8

Let *f(x)* and *g(x)* be functions that are continuous on the interval $-2 \le x \le 5$ and that satisfy

$$\int_{-2}^{5} f(x)dx = 3 \quad \int_{-2}^{5} g(x)dx = -4 \quad \int_{3}^{5} g(x)dx = 7$$

Use this information to evaluate each of these definite integrals:

a. $\int_{-2}^{5} [2f(x) - 3g(x)]dx$ b. $\int_{-2}^{3} f(x)dx$

$$\int_{0}^{1} 8x(x^2+1)^3 dx$$

$$\int_{1/4}^{2} \left(\frac{\ln x}{x} \right) dx$$

Net Change

If $Q'(x)$ is continuous on the interval $a \leq x \leq b$, then the **net change** in $Q(x)$ as x varies from $x=a$ to $x=b$ is given by

$$Q(b) - Q(a) = \int_a^b Q'(x)dx$$

Example 5.3.9

At a certain factory, the marginal cost is $3(q-4)^2$ dollars per unit when the level of production is q units.

By how much will the total manufacturing cost increase if the level of production is raised from 6 units to 10 units?

Example 5.3.10

A protein with mass m (grams) disintegrates into amino acids at a rate given by $\dfrac{dm}{dt} = \dfrac{-30}{(t+3)^2}$ g/hr

What is the net change in mass of the protein during the first 2 hours?

Area justification of the FTC (p409-410)

5.4 Applying Definite Integration: Area Between Curves and Average Value

A procedure for using definite integration in application (p414)

The Area Between Two Curves

- If $f(x)$ and $g(x)$ are continuous with $f(x) \geq g(x)$ on the interval $a \leq x \leq b$, then the area A between the curves $y = f(x)$ and $y = g(x)$ over the interval is given by

$$A = \int_a^b [f(x) - g(x)]dx$$

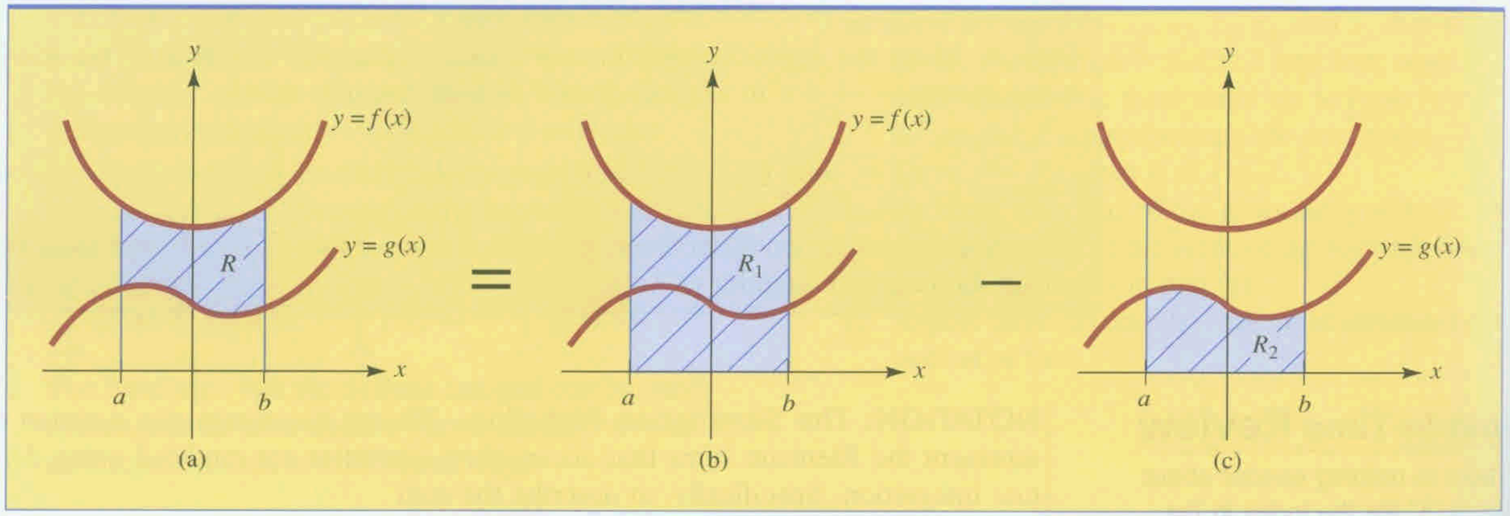


FIGURE 5.9 Area of R = area of R_1 − area of R_2.

Notes

- The formula for the area between two curves could be applied to the $f(x)<0$ and $g(x)<0$, but $f(x) \geqq g(x)$.
- The procedure for applying definite integration is seen on p416-417.

Example 5.4.1

Find the area of the region R enclosed by the curves $y = x^3$ and $y = x^2$

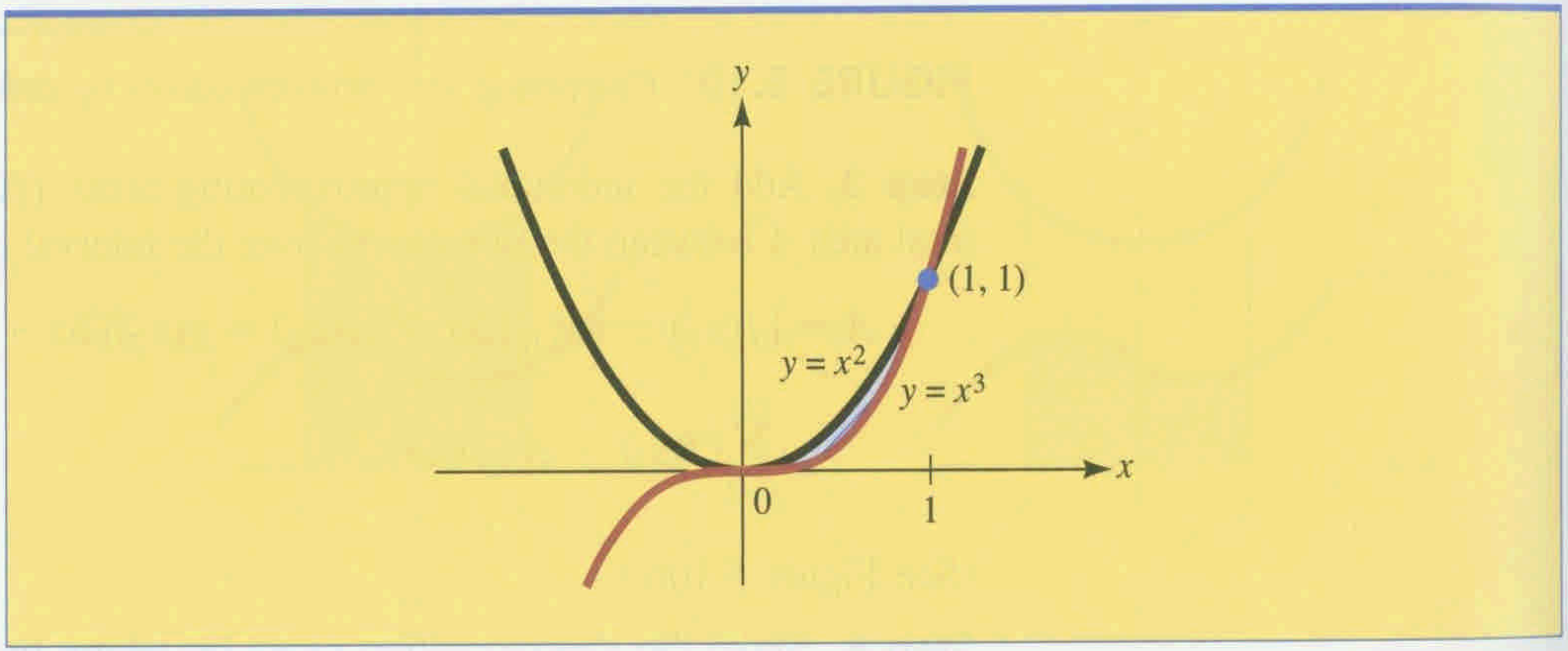


FIGURE 5.11 The region enclosed by the curves $y = x^2$ and $y = x^3$.

Example 5.4.2

Find the area of the region enclosed b the line $y= 4x$ and the curve $y= x^3 +3x^2$

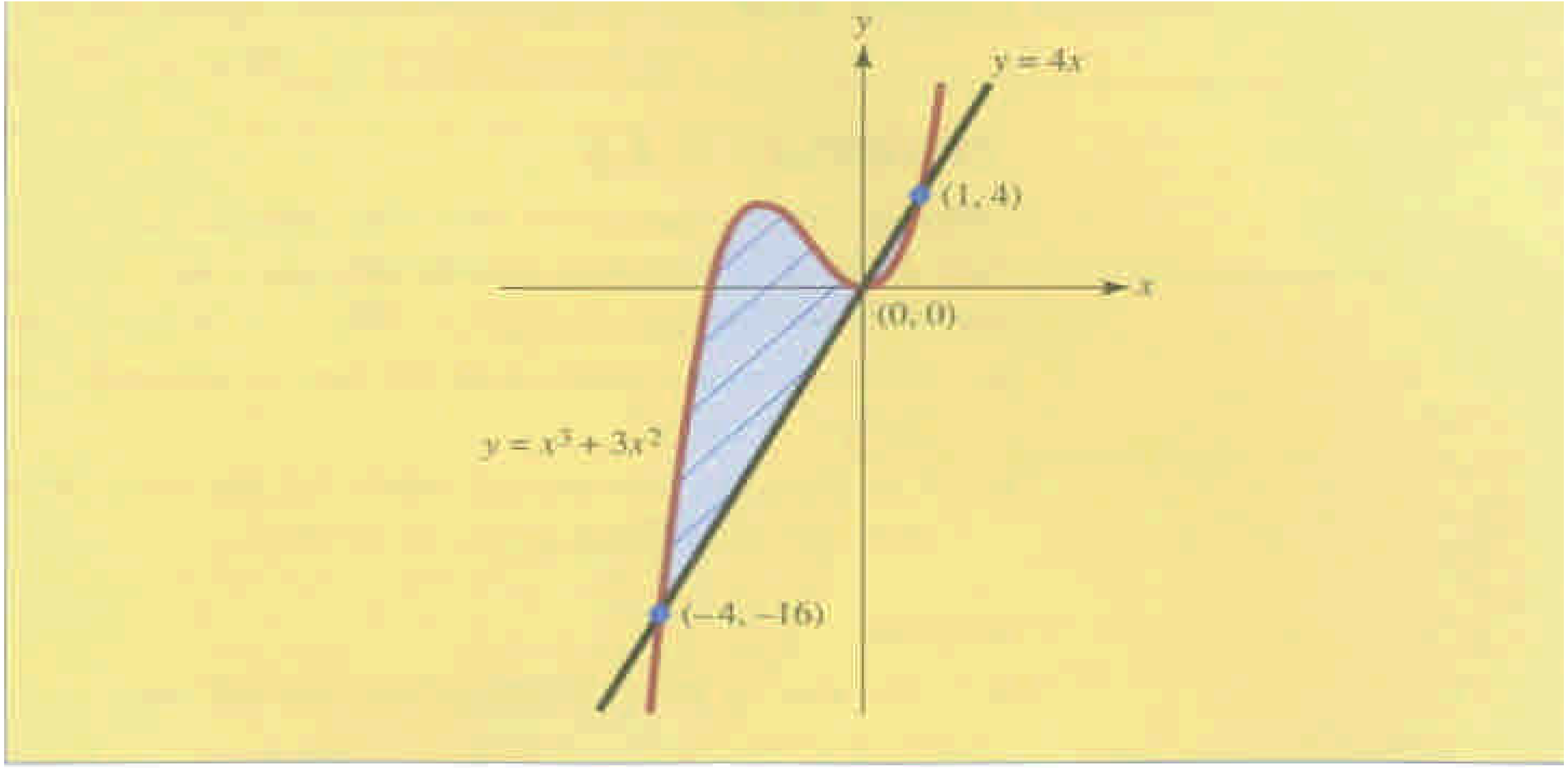


FIGURE 5.12 The region enclosed by the line $y = 4x$ and the curve $y = x^3 + 3x^2$.

Net Excess Profit

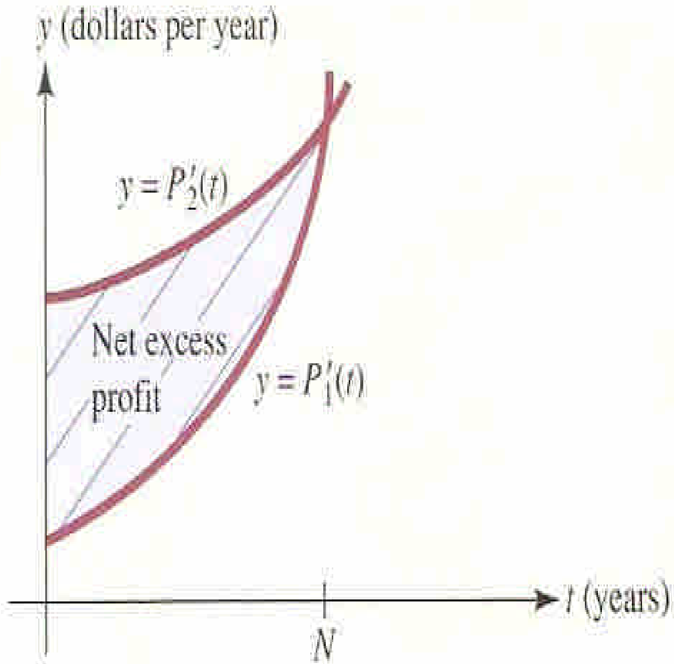


FIGURE 5.13 Net excess profit as the area between rate of profitability curves.

Example 5.4.3

- Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t)=50+t^2$ hundred dollars per year, while a second investment will be generating profit at the rate of $P_2'(t)=200+5t$ hundred dollars per year.

a. For how many years does the rate of profitability of the second investment exceed that of the first?
b. Compute the net excess profit for the time period determined in part (a). Interpret the net excess profit as an area.

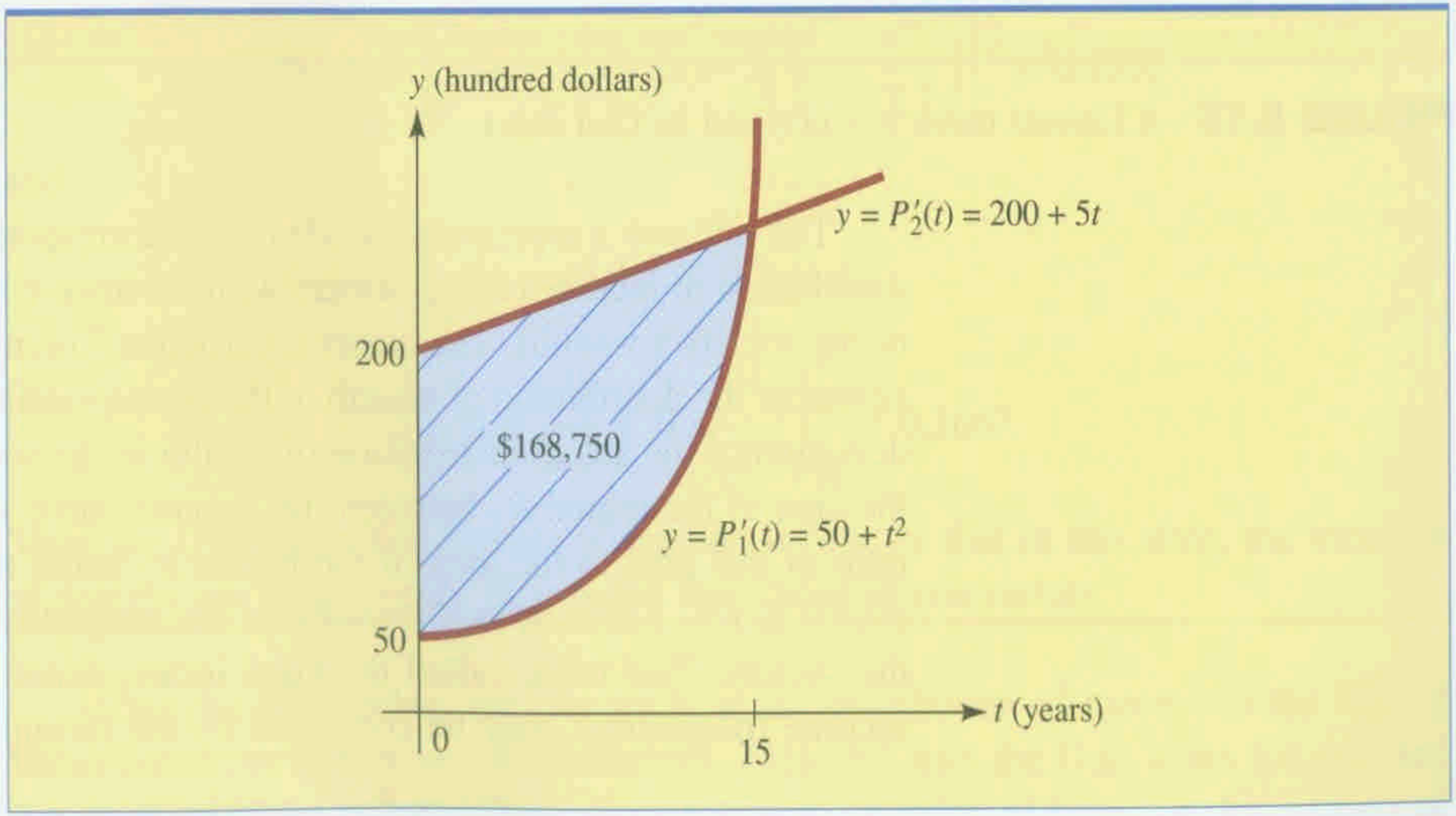


FIGURE 5.14 Net excess profit for one investment plan over another.

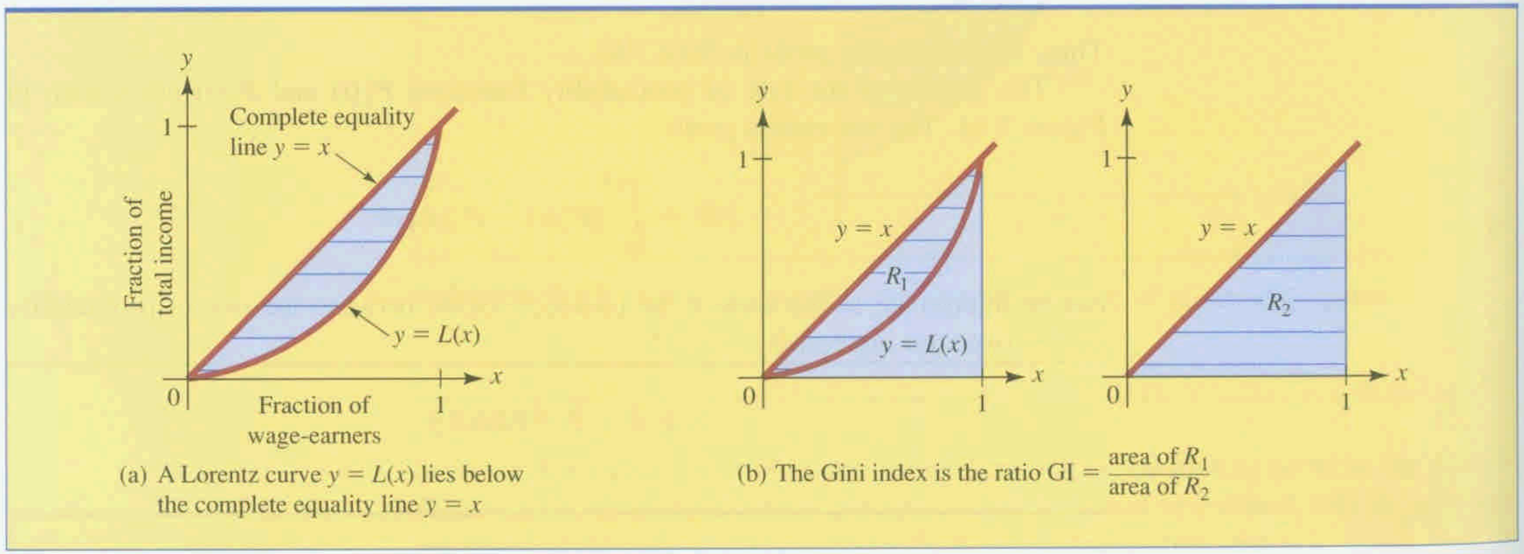


FIGURE 5.15 A Lorentz curve $y = L(x)$ and its Gini index.

Gini Index (index of incoming inequality)

If $y=L(x)$ is the equation of a Lorentz curve, then the inequality in the corresponding distribution of wealth is measured by the Gini index, which is given by the formula

$$Gini\ index = 2\int_0^1 [x - L(x)]dx$$

Example 5.4.4

A governmental agency determines that the Lorentz curves for the distribution of income for dentists and contractors in a certain state are given by the functions

$$L_1(x) = x^{1.7} \quad \text{and} \quad L_2(x) = 0.8x^2 + 0.2x$$

respectively.

For which profession is the distribution of income more fairly distributed?

Table 5.1 Gini Indices for Selected Countries

Country	Gini Index
United Sates	0.460
Brazil	0.601
Canada	0.315
Denmark	0.247
Germany	0.281
Japan	0.350
South Africa	0.584
Panama	0.568
Thailand	0.462
United Kingdom	0.326

SOURCE: David C. Colander, Economics, 4th ed., Boston: McGraw-Hill, 2001, p. 435

The Average Value of a Function

Let $f(x)$ be a function that is continuous on the interval $a \leq x \leq b$, Then the average value V of $f(x)$ over $a \leq x \leq b$ is given by the definite integral

$$V = \frac{1}{b-a} \int_a^b f(x)\,dx$$

Example 5.4.5

A manufacturer determines that t months after introducing a new product, the company's sales will be *S(t)* thousand dollars, where

$$S(t) = \frac{750t}{\sqrt{4t^2 + 25}}$$

What are the average monthly sales of the company over the first 6 months after the introduction of the new product?

Example 5.4.6

A researcher models the temperature T (in ^{0}C) during the time period from 6 A.M. to 6 P.M. in a certain northern city by the function

$$T(t) = 3 - \frac{1}{3}(t-4)^2 \text{ for } 0 \le t \le 12$$

where t is the number of hours after 6 A.M.

a. What is the average temperature in the city during the workday, from 8 A.M. to 5 P.M.?
b. At what time (or times) during the workday is the temperature in the city the same as the average temperature found in part(a)?

5.5 Additional Applications to Business and Economics

- **Useful Life of a Machine**
  - A manager may consider disposing of the machine at the time T when $C'(T)=R'(T)$
  - the time period $0 \le t \le T$ is called the **useful life** of the machine

Example 5.5.1

Suppose that when it is t years old, a particular industrial machine is generating revenue at the *rate* $R'(t)=5{,}000-20t^2$ dollars per year and that operating and servicing costs related to the machine are accumulating at the rate $C'(t)=2{,}000+10t^2$ dollars per year.

a. What is the useful life of this machine?

b. Compute the net profit generated by the machine over its period of useful life.

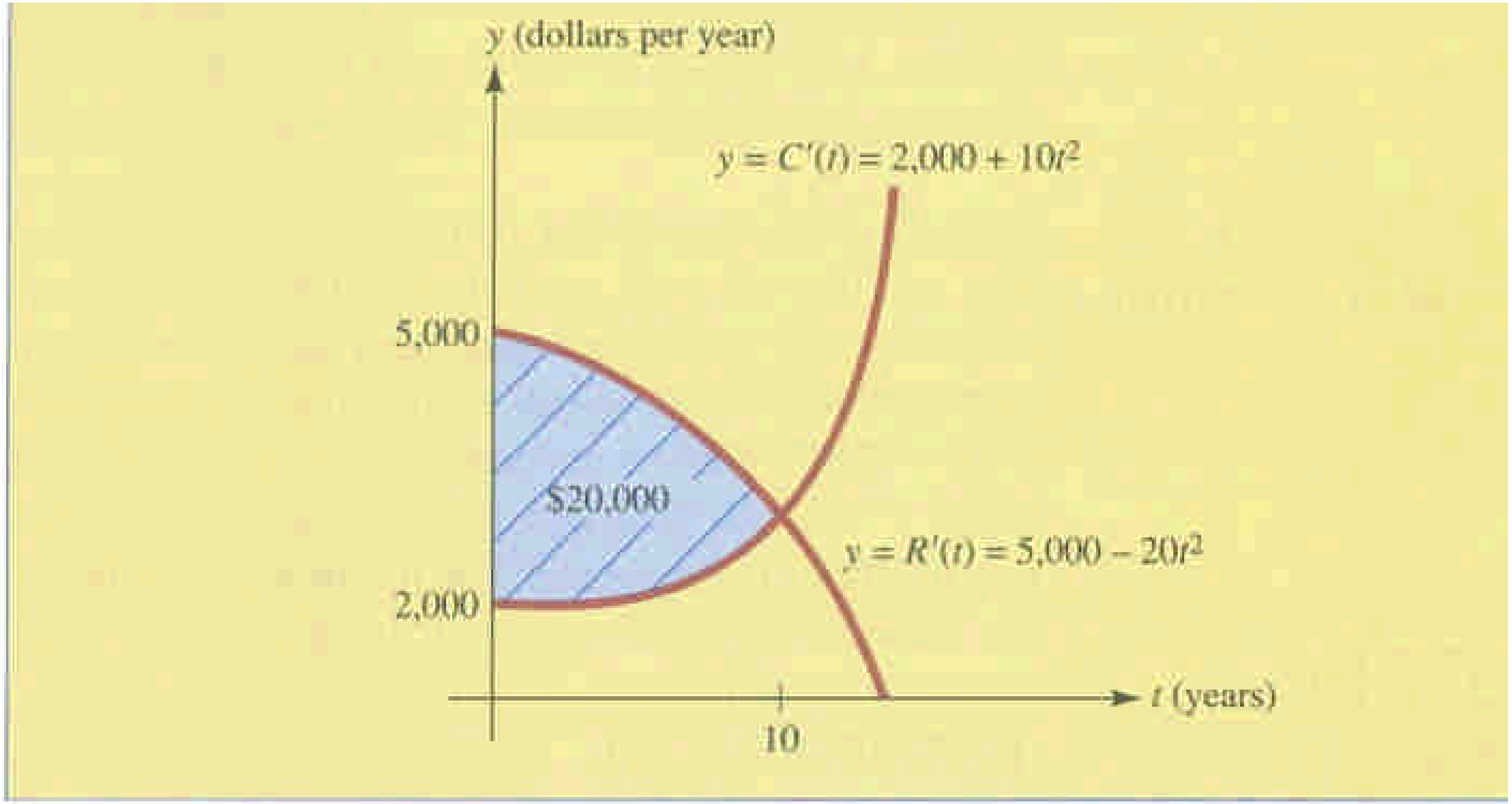


FIGURE 5.17 Net profit over the useful life of a machine.

Example 5.5.2

Money is transferred continuously into an account at the constant rate of \$1,200 per year.
The account earns interest at the annual rate of 8% compounded continuously.

How much will be in the account at the end of 2 years?

Example 5.5.3

Jane is trying to decide between two investments. The first costs \$1,000 and is expected to generate a continuous income stream at the rate of $f_1(t)=3{,}000e^{0.03t}$ dollars per year.
The second investment costs \$4,000 dollars and is estimated to generate income at the constant rate 4000 per year.

If the prevailing annual interest rate remains fixed at 5% compounded continuously over the next 5 years, which investment is better over this time period?

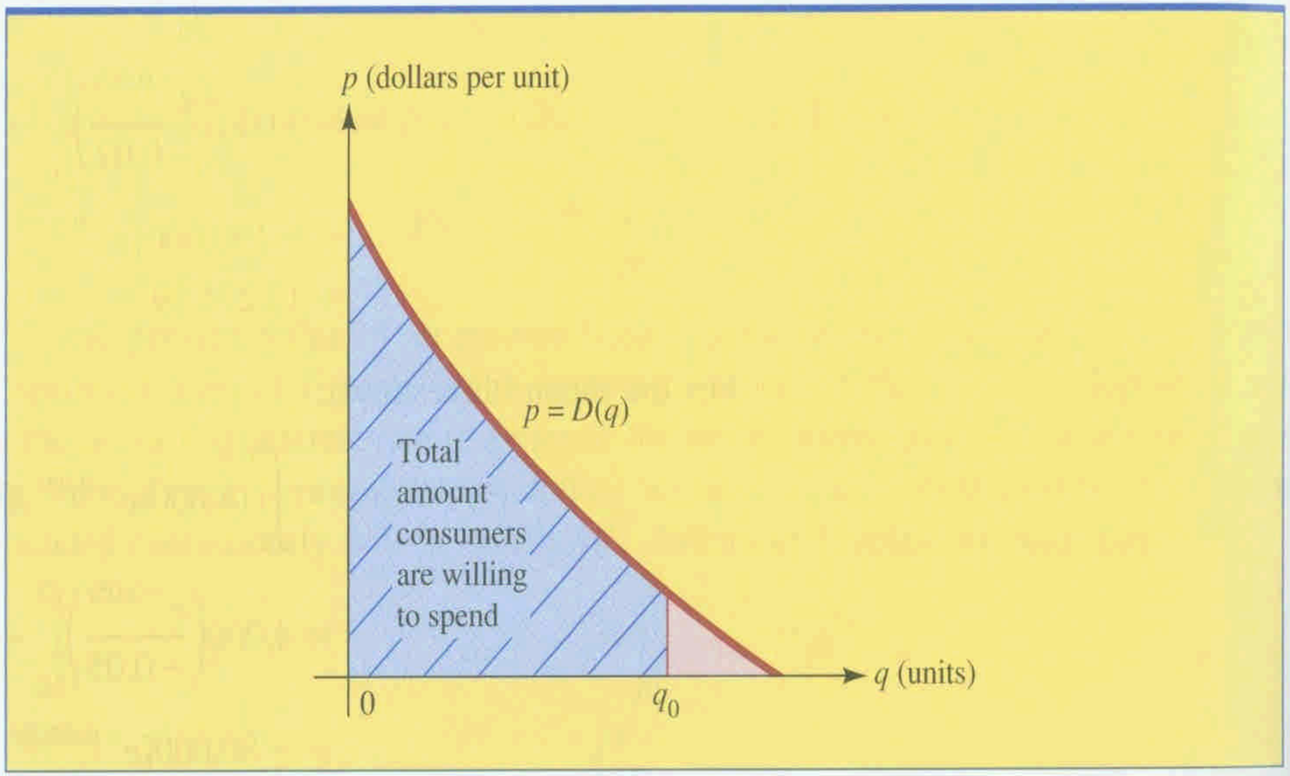


FIGURE 5.19 The amount consumers are willing to spend is the area under the demand curve.

Example 5.5.4

Suppose that the consumers' demand function for a Certain commodity is $D(q)=4(25-q^2)$ dollars per unit.

a. Find the total amount of money consumers are willing to spend to get 3 units of the commodity.
b. Sketch the demand curve and interpret he answer to part (a) as an area.

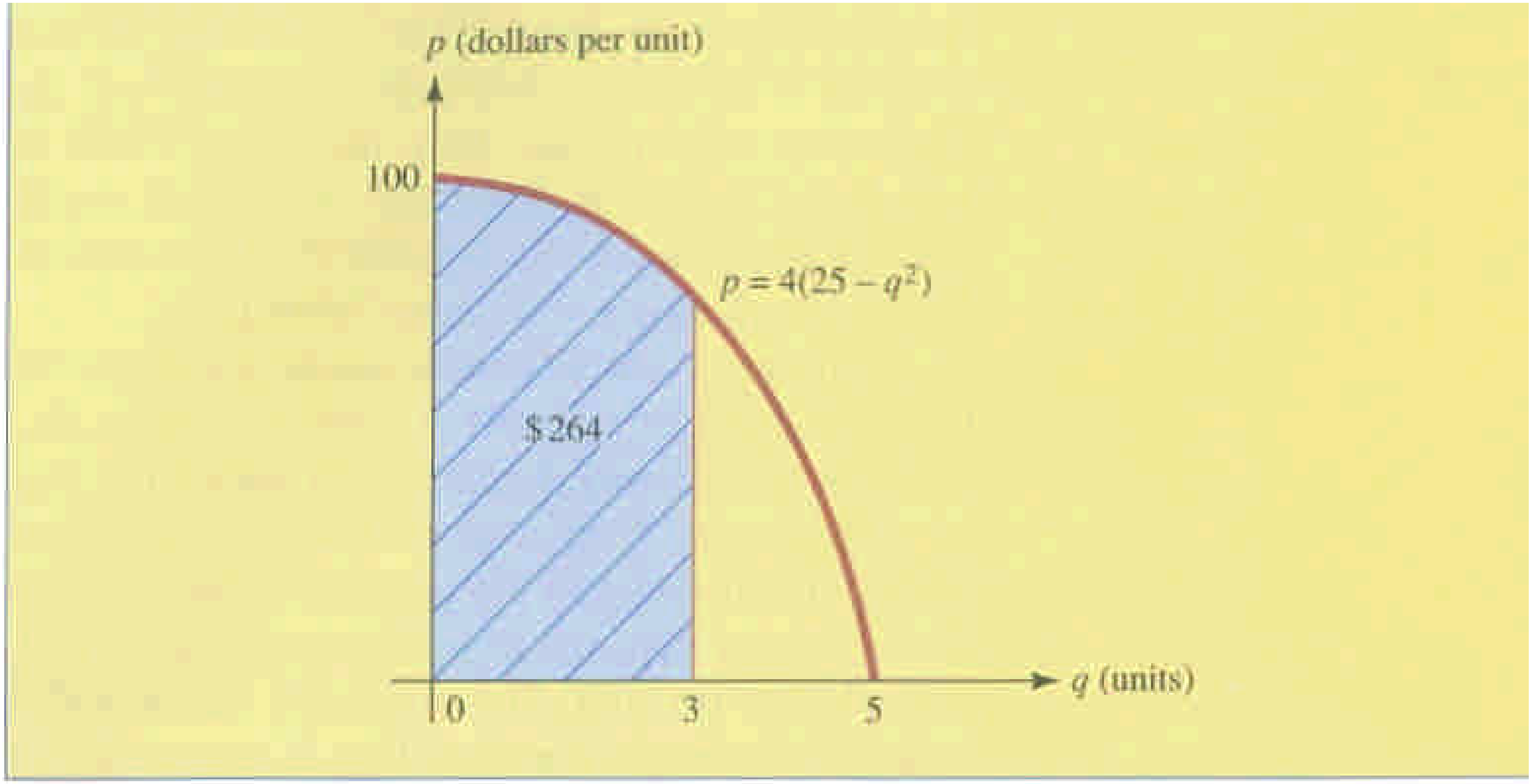


FIGURE 5.20 Consumers' willingness to spend for 3 units when demand is given by $D(q) = 4(25 - q^2)$.

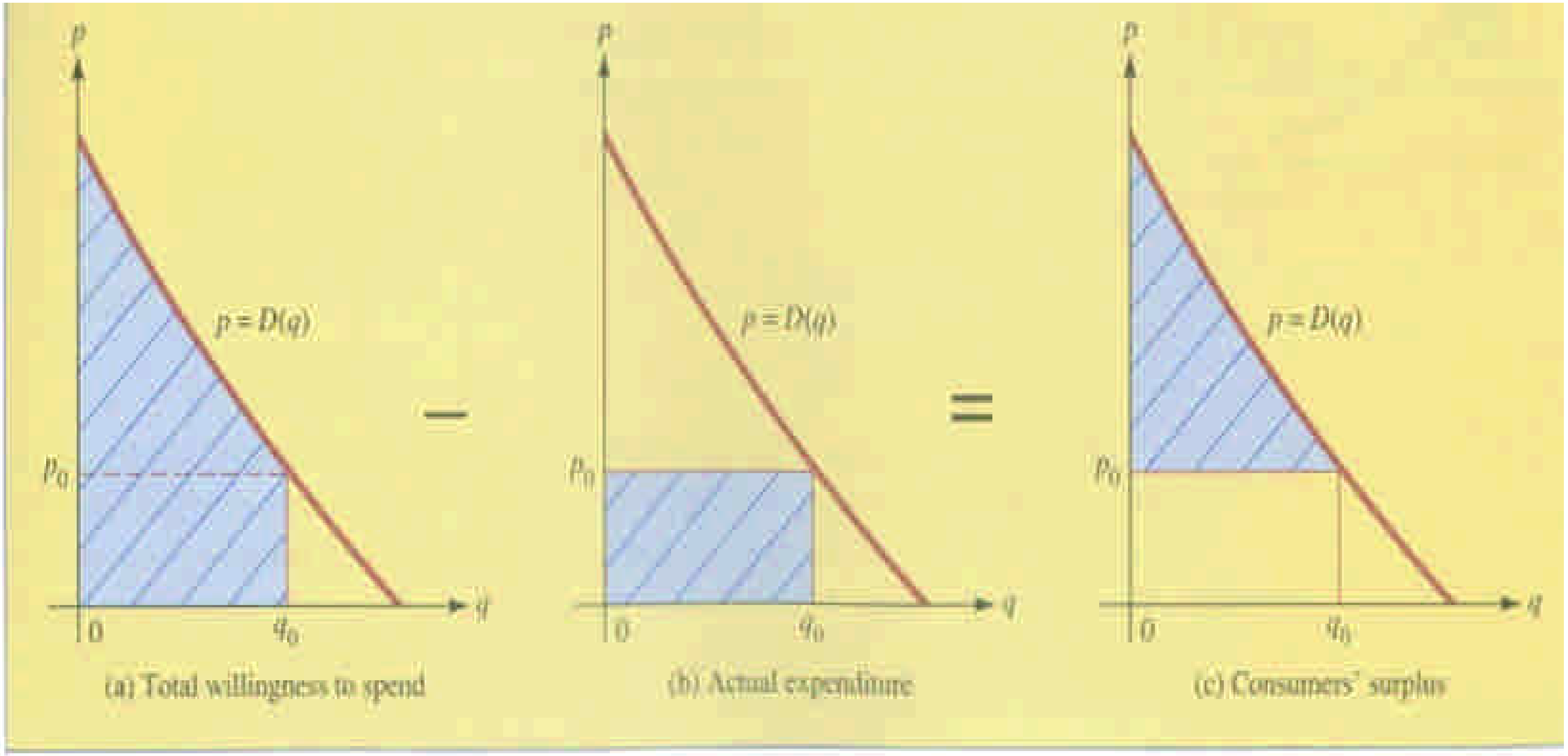


FIGURE 5.21 Geometric interpretation of consumers' surplus.

Example 5.5.5

A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is $p = D(q) = -0.1q^2 + 90$

dollars per tire, and the same number of tires will be supplied when the price is

$$p = S(q) = -0.2q^2 + q + 50$$

dollars per tire.

a. Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price.
b. Determine the consumers' and producers' surplus at the equilibrium price.

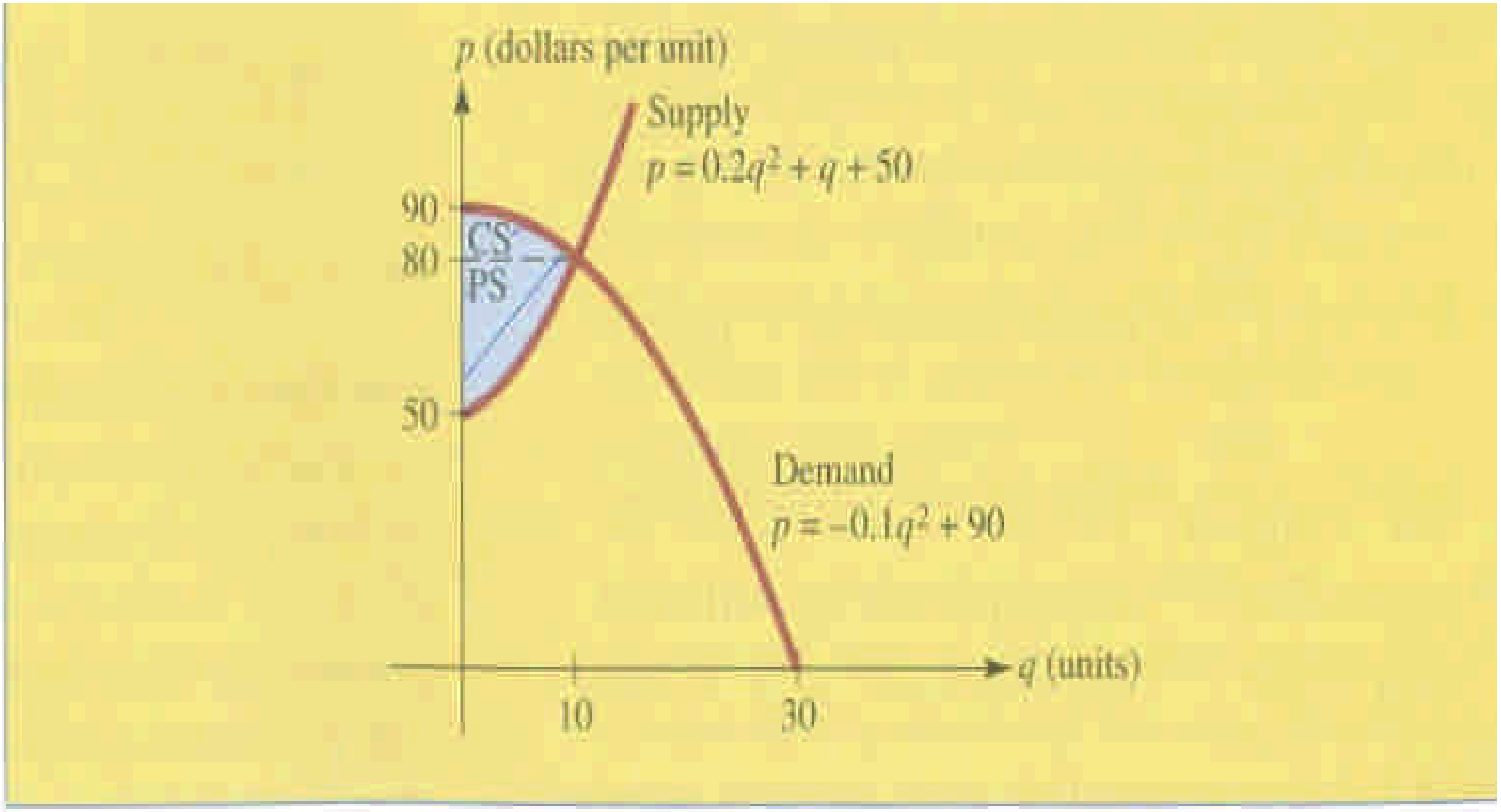


FIGURE 5.22 Consumers' surplus and producers' surplus for the demand and supply functions in Example 5.5.5.

5.6 Additional Applications to the Life and Social Sciences

Example 5.6.1(1/3)

- A new county mental health clinic has just opened.
- Statistics from similar facilities suggest that the fraction of patients who will still be receiving treatment at the clinic t months after their initial visit is given by the function $f(t) = e^{-t/20}$.
- The clinic initially accepts 300 people for treatment and plans to accept new patients at the constant rate of $g(t)=10$ patients per month.
- **Q**: Approximately how many people will be receiving treatment at the clinic 15 months from now?

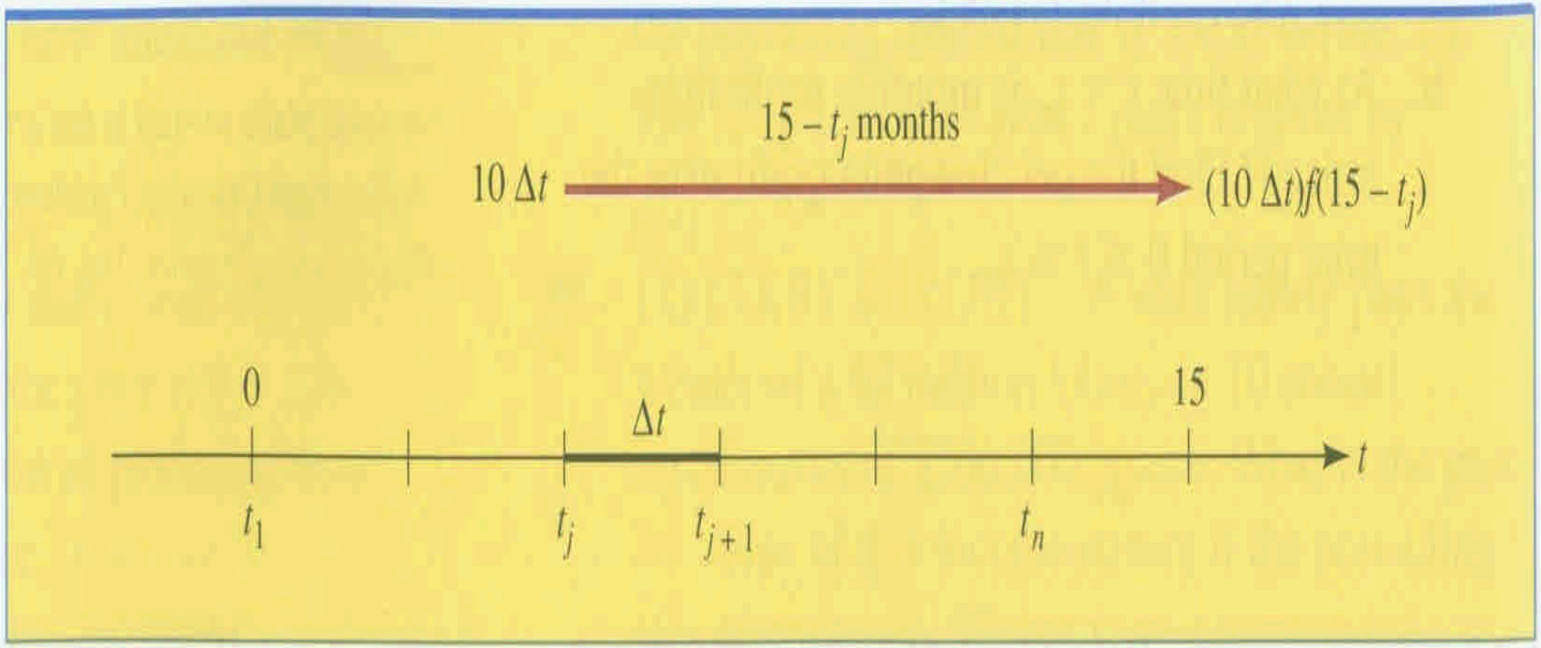


FIGURE 5.23 New members arriving during the jth subinterval.

Example 5.6.2

- A mild toxin is introduced to a bacterial colony whose current population is 600,000.
- Observations indicate that $R(t)=200\,e^{0.01t}$ bacteria per hour are born in the colony at time t and that the fraction of the population that survives for t hours after birth is $S(t)=e^{-0.015t}$.
- **Q**: What is the population of the colony after 10 hours?

Example 5.6.3

- Find an expression for the rate (in cubic centimeters per second) at which blood flows through an artery of radius R if the speed of the blood r centimeters from the central axis is $S(r)= k(R^2\text{-}R^2)$, where k is a constant.

Example 5.6.4

- A physician injects 4 mg of dye into a vein near the heart of a patient, and a monitoring device records the concentration of dye in the blood at regular intervals over a 23-second period.
- It is determined that the concentration at time t $(0 \leq t \leq 23)$ is closely approximated by the function.
- **Q**: Based on this information, what is the patient's cardiac output?

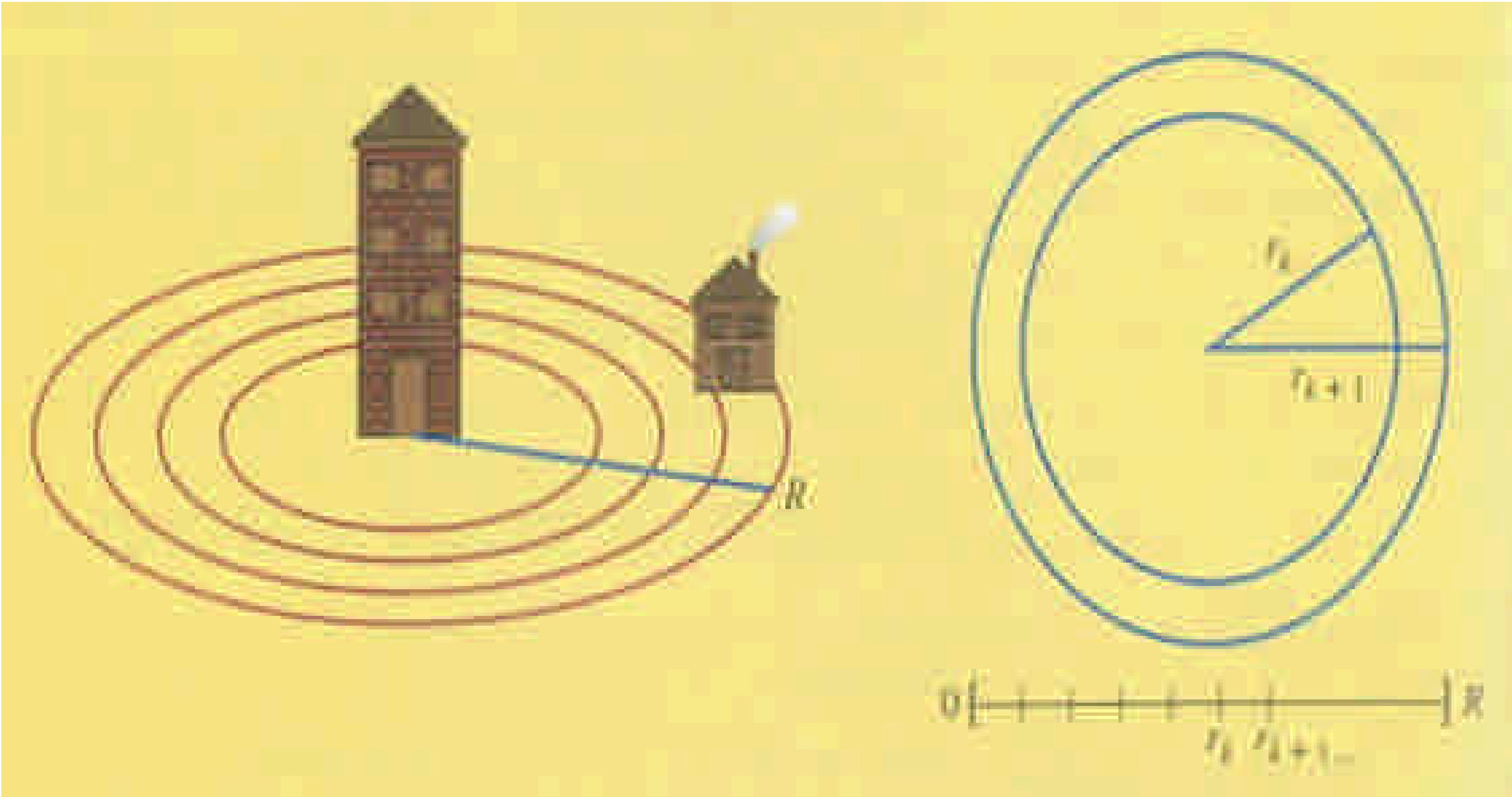
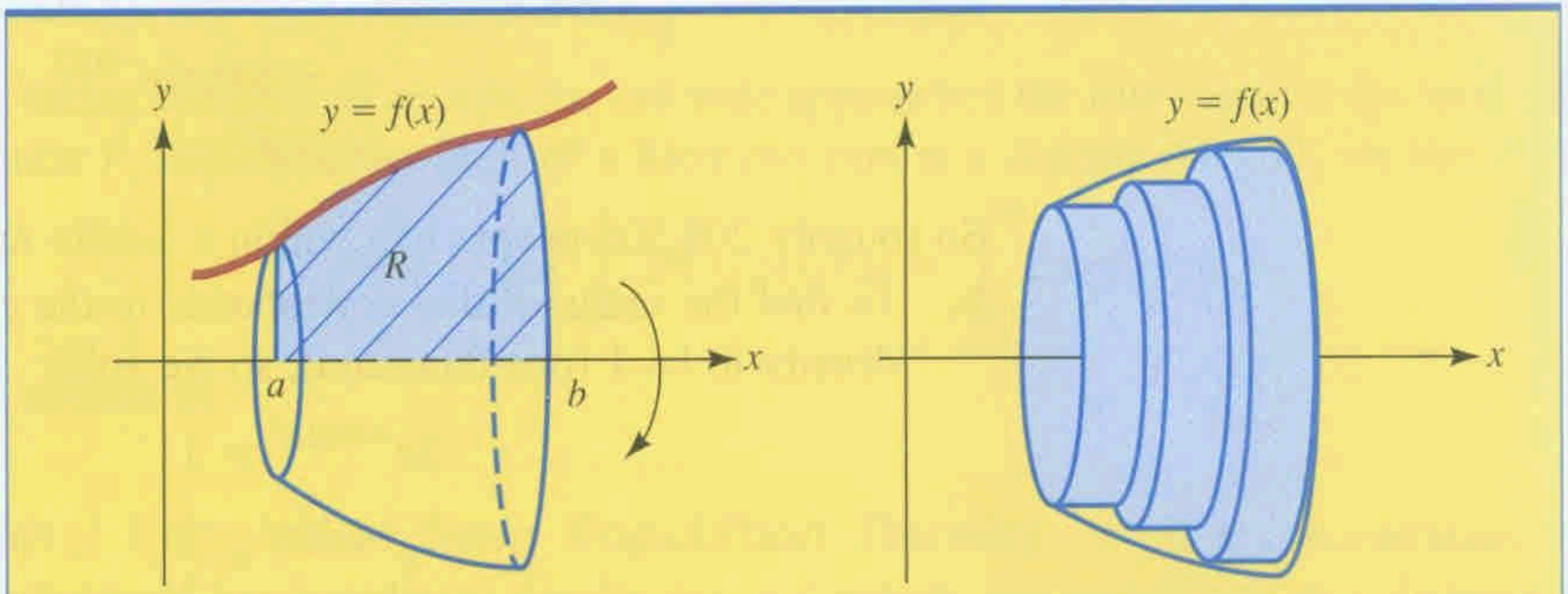


FIGURE 5.26 Subdividing an urban area into concentric rings.

Example 5.6.5

- A city has population *density* $p(r) = 3e^{-0.01r^2}$, where $p(r)$ is the number of people (in thousands) per square mile at a distance of r miles from the city center.
  a. What population lives within a 5-mile radius of the city center?
  b. The city limits are set at a radius R where the population density is 1,000 people per square mile. What is the total population within the city limits?



(a) The region R under $y = f(x)$ between $x = a$ and $x = b$ is rotated about the x axis to generate the solid S.

(b) The approximation of R by rectangles and the corresponding approximation of S by disks.

FIGURE 5.27 A solid S formed by rotating the region R about the x axis.

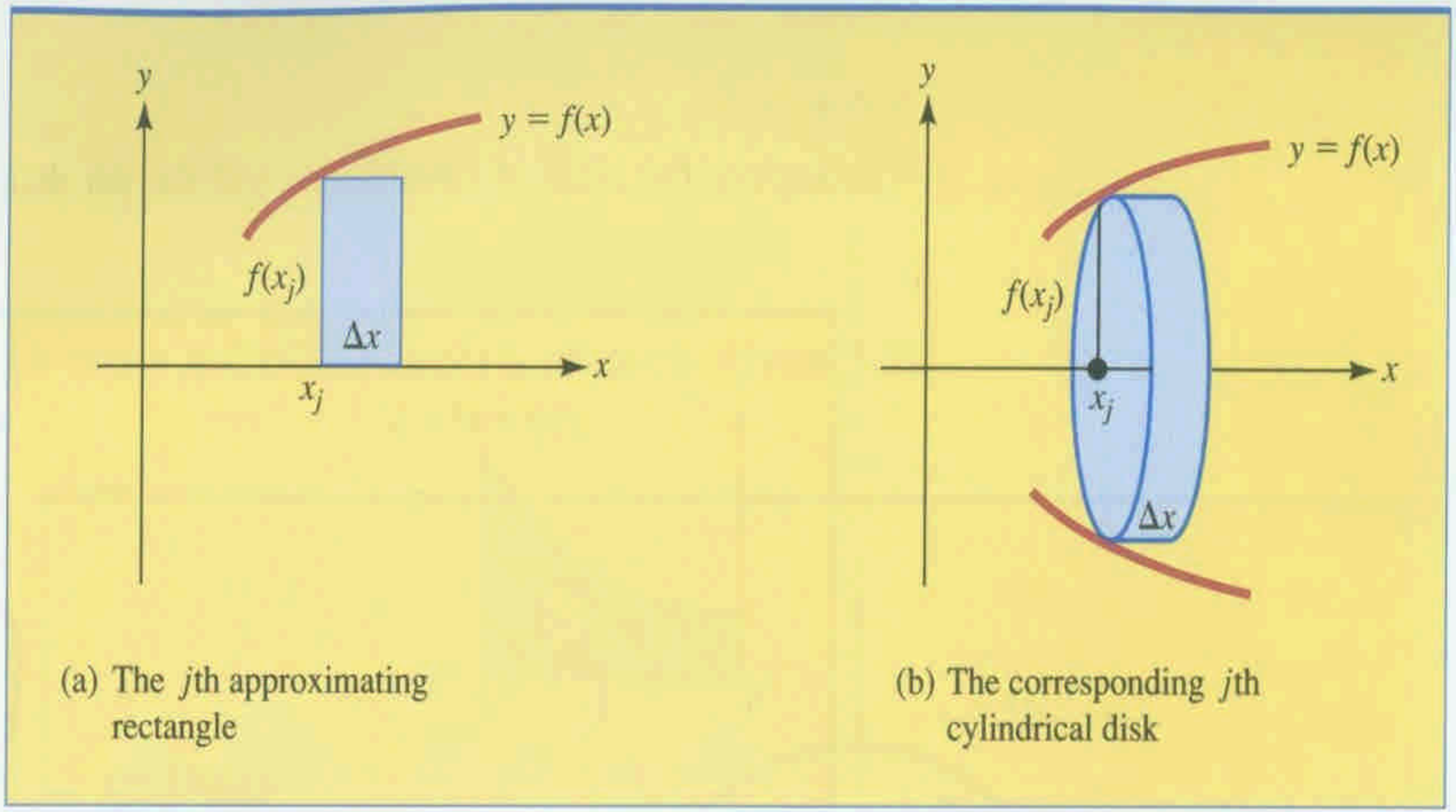


FIGURE 5.28 The volume of the solid S is approximated by adding volumes of approximating disks.

Example 5.6.6

Find the volume of the solid S formed by revolving the region under the curve $y=x^2+1$ from $x=0$ to $x=2$ about the x axis.

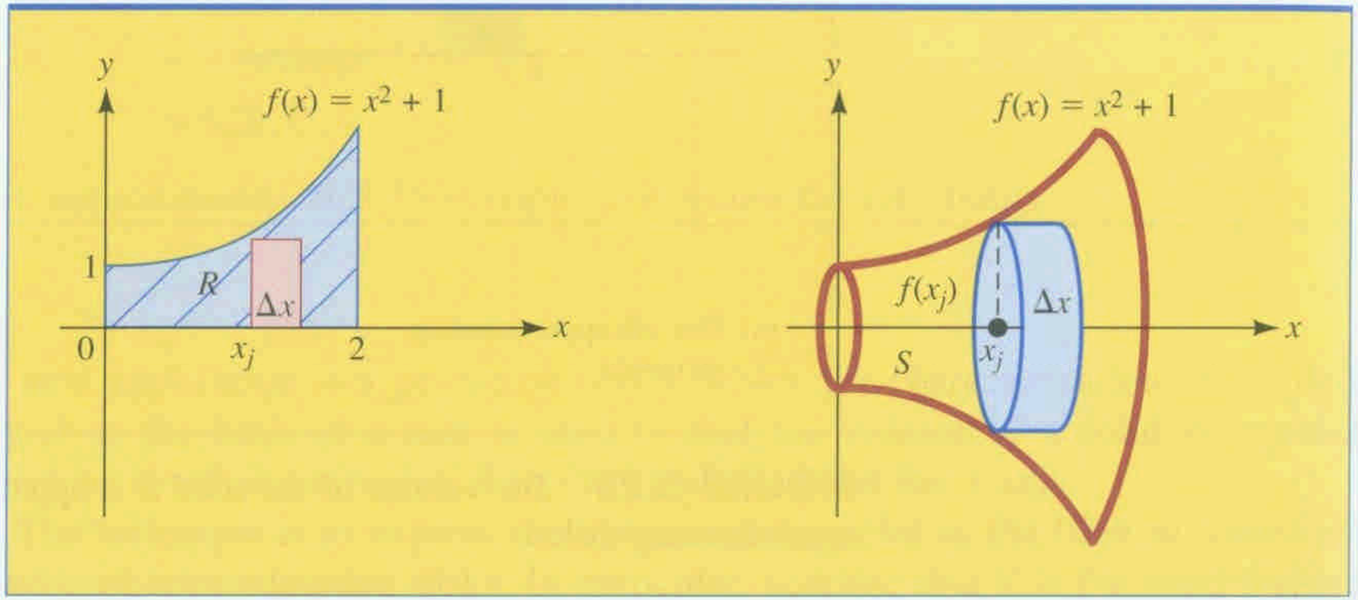


FIGURE 5.29 The solid formed by rotating about the x axis the region under the curve $y = x^2 + 1$ between $x = 0$ and $x = 2$.

Example 5.6.7

A tumor has approximately the same shape as the solid formed by rotating the region under the curve

$$y = \frac{1}{3}\sqrt{16 - 4x^2}$$

about the x axis, where x and y are measured in centimeters. Find the volume of the tumor.

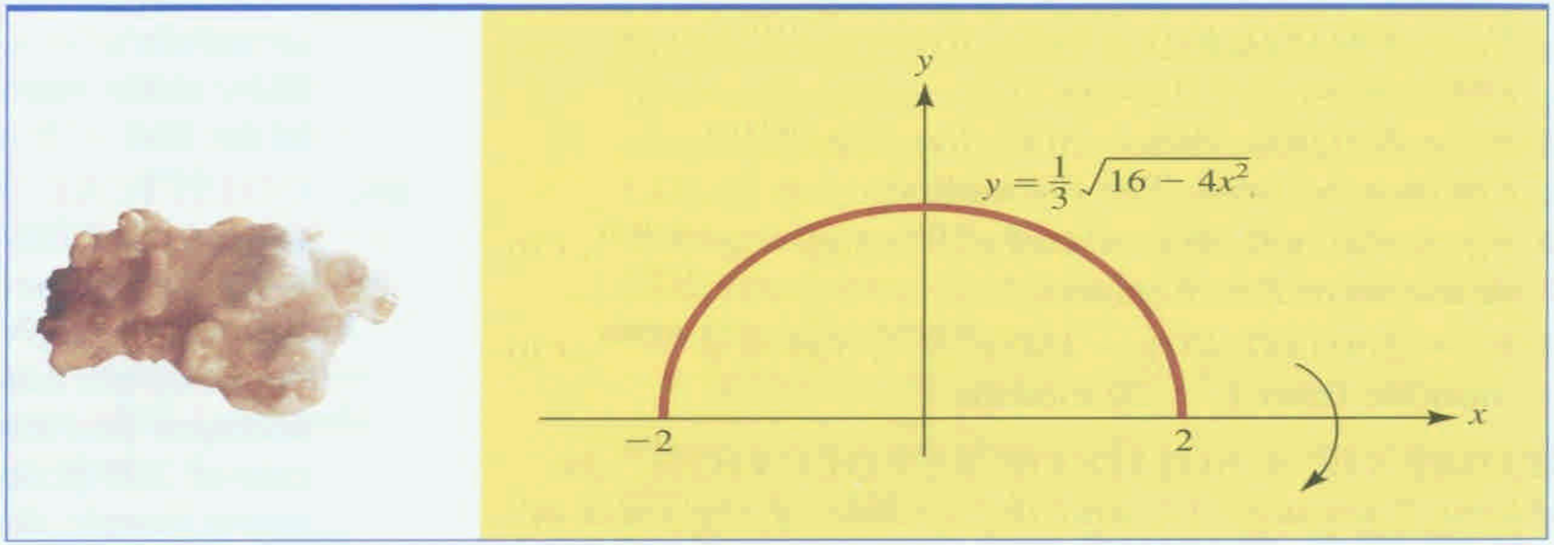


FIGURE 5.30 Tumor with the shape of the solid formed by rotating the curve $y = \frac{1}{3}\sqrt{16 - 4x^2}$ about the x axis.