CHAPTER 6
Additional Topics in Integration

- Integration by Parts; Integral Tables
- Introduction to Differential Equations
- Improper Integrals; Continuous Probability
- Numerical Integration

6.1 Integration By Parts (IBP, 分部積分)

- The derivation of the formula of IBP, see p476.
- Practice Example 6.1.1, find $\int x^2 \ln xdx$
- Procedure for IBP, see p477.
- Q1: What is the timing for the use of IBP?
- Q2: What are items included in IBP?
- Q3: What are the characteristics of IBP?

Example 6.1.2-3

Find $$\int xe^{2x}dx$$

$$\int x\sqrt{x+5}dx$$

Definite IBP, see p479.

Example 6.1.4

Find the area of the region bounded by the curve y = ln$x$, the x axis, and the lines $x = 1$ and $x = e$.

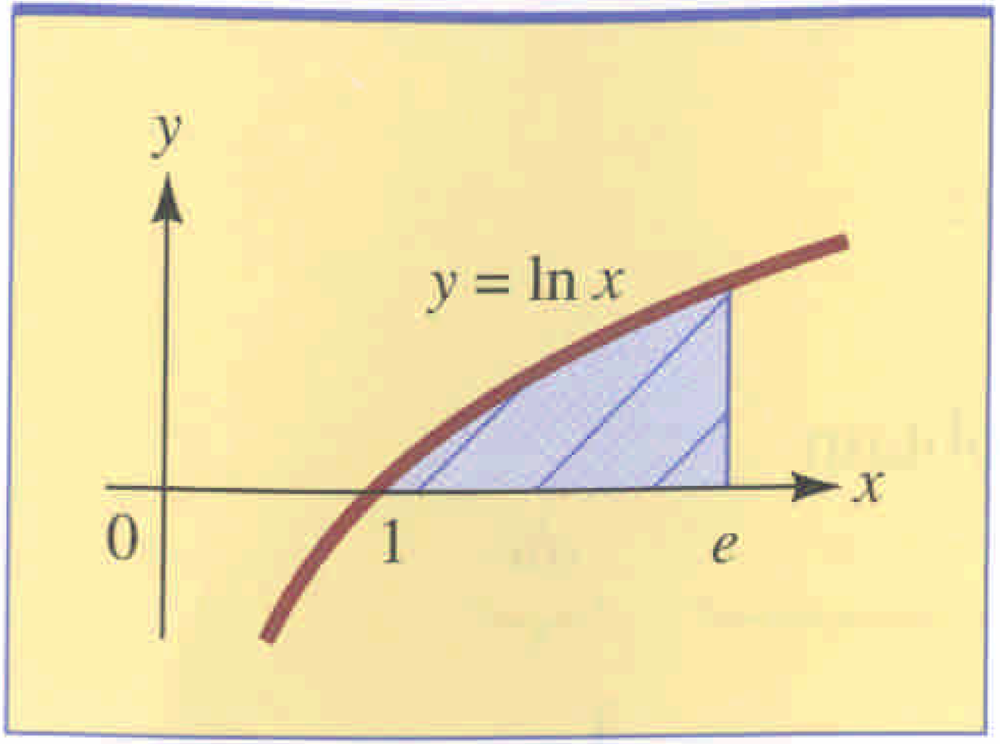

FIGURE 6.1 The region under $y = \ln x$ over $1 \leq x \leq e$.

Example 6.1.5

- Joyce is considering a 5-year investment and estimates that t years from now it will be generating a continuous income stream of 3,000+50t dollars per year.
- If the prevailing annual interest rate remains fixed at 4% compounded continuously during the entire 5-year term, what should the investment be worth in 5 years?

Example 6.1.6-11

Find $\int x^2 e^{2x}dx$ $\int x^3 e^{5x}dx$

$$\int \frac{1}{x(3x-6)}dx$$

$$\int \frac{1}{6-3x^2}dx$$

$$\int \frac{1}{3x^2+6}dx$$

$$\int \frac{1}{\sqrt{4x^2-9}}dx$$

Table 6.1: A short table of integrals (p484)

SECTION 6.2
Introduction to Differential Equations

- Recall a dynamic process involving 3 stages in Section 1.4
- Sometimes the mathematical formulation of a problem involves an equation in which
  - a quantity and the rate of change of that quantity are related by an equation.
- Since rates of change are expressed in terms of derivatives or differentials, such an equation is approximately called a differential equation (DE).
  - Examples, see p491.

SECTION 6.2
Introduction to Differential Equations

- DE are among the most useful tools for modeling continuous phenomena, such as
  - population dynamics, chemical kinetics, spread of disease, dynamic economic behavior, ecology, and the transmission (傳送) of information.
- The simplest type of DE has the form, $dy/dx=g(x)$, in which
  - the derivative of the quantity y is given explicitly as a function of the independent variable x.

SECTION 6.2
Introduction to Differential Equations

- A complete characterization of all possible solutions of the equation is called a **general solution**.
- A differential equation coupled with (連接) a side condition is referred to as an **initial value problem**, and a solution that satisfies both the differential equation and the side condition is called a **particular solution** of the initial value problem.

Example 6.2.1

Find the general solution of the differential equation

$$\frac{dy}{dx} = x^2 + 3x$$

and the particular solution that satisfies y = 2 when x = 1.

Example 6.2.2

- The resale value of a certain industrial machine decrease over a 10-year period at a rate that depends on the age of the machine.

- When the machine is x years old, the rate at which its value is changing is $220(x - 10)$ dollars per year.

- Q1: Express the value of the machine as function of its age and initial value.
- Q2: If the machine was originally worth \$12,000, how much will it be worth when it is 10 years old ?

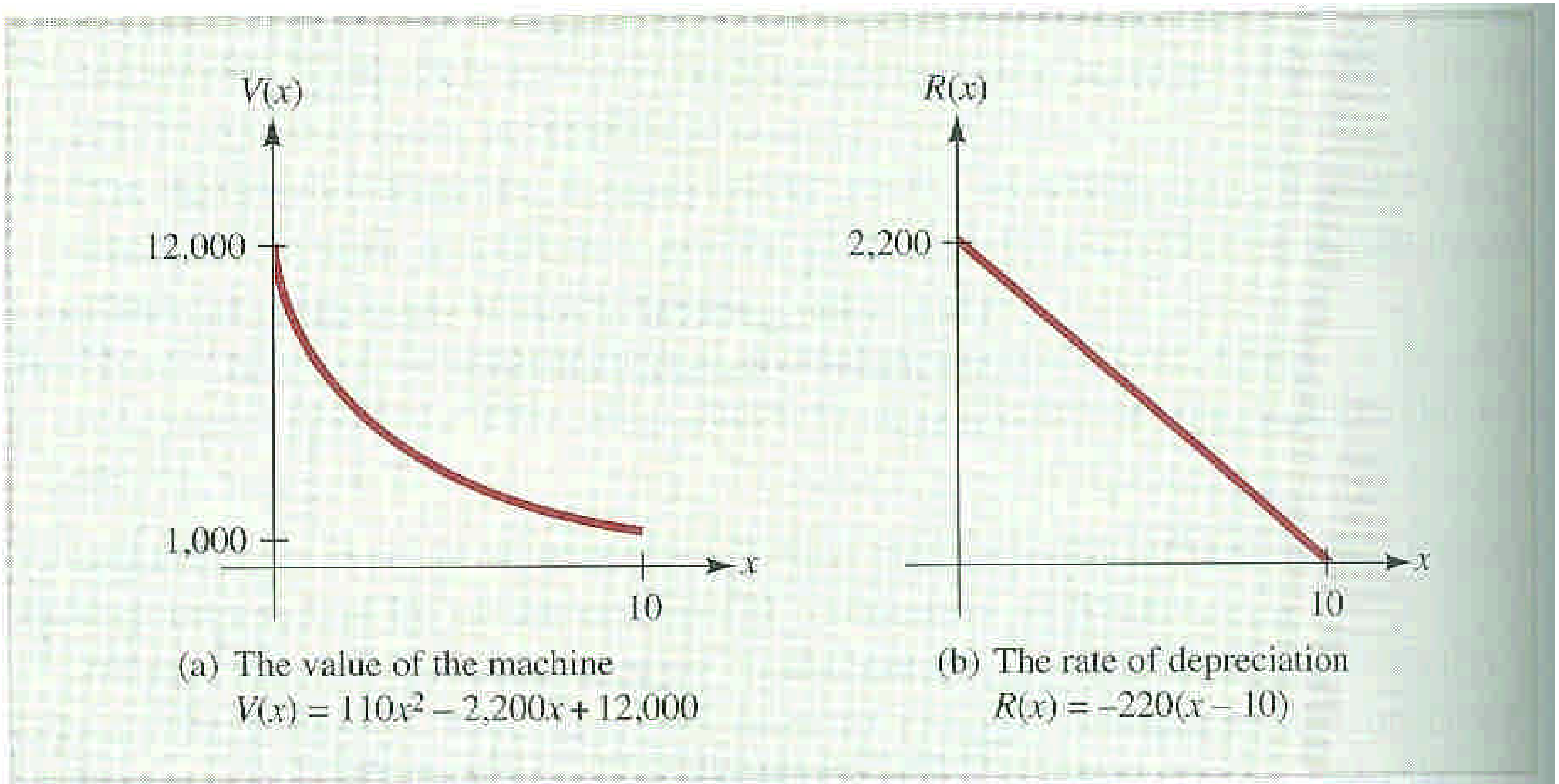

FIGURE 6.2 The value of the machine and its rate of depreciation.

Example 6.2.3

- An oil well that has just been opened is expected to yield 300 barrels of crude oil per month and, at that rate, is expected to run dry in 3 years.

- It is estimated that t months from now, the price of crude oil will be $P(t) = 28 + 0.3\sqrt{t}$ dollars per barrel.

- Q: If the oil is sold as soon as it is extracted from the ground, what is the total revenue generated by the well during its operation?

Separable differential equations (p494)

Example 6.2.4

Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{2x}{y^2}$$

Example 6.2.5

An object moves along the x axis in such a way that at each time *t*, its velocity is given by the differential equation

$$\frac{dx}{dt} = x^2 \ln t$$

If the object is at x = -2 when t = 1, where is it when t = 3?

Exponential growth and decay (P495)
Example 6.2.6

Example 6.2.7 (learning model, p496)

- The rate at which people hear about a new increase in postal rates is proportional to the number of people in the country who have not yet heard about it.

- Q: Express the number of people who have heard already about the increase as a function of time.

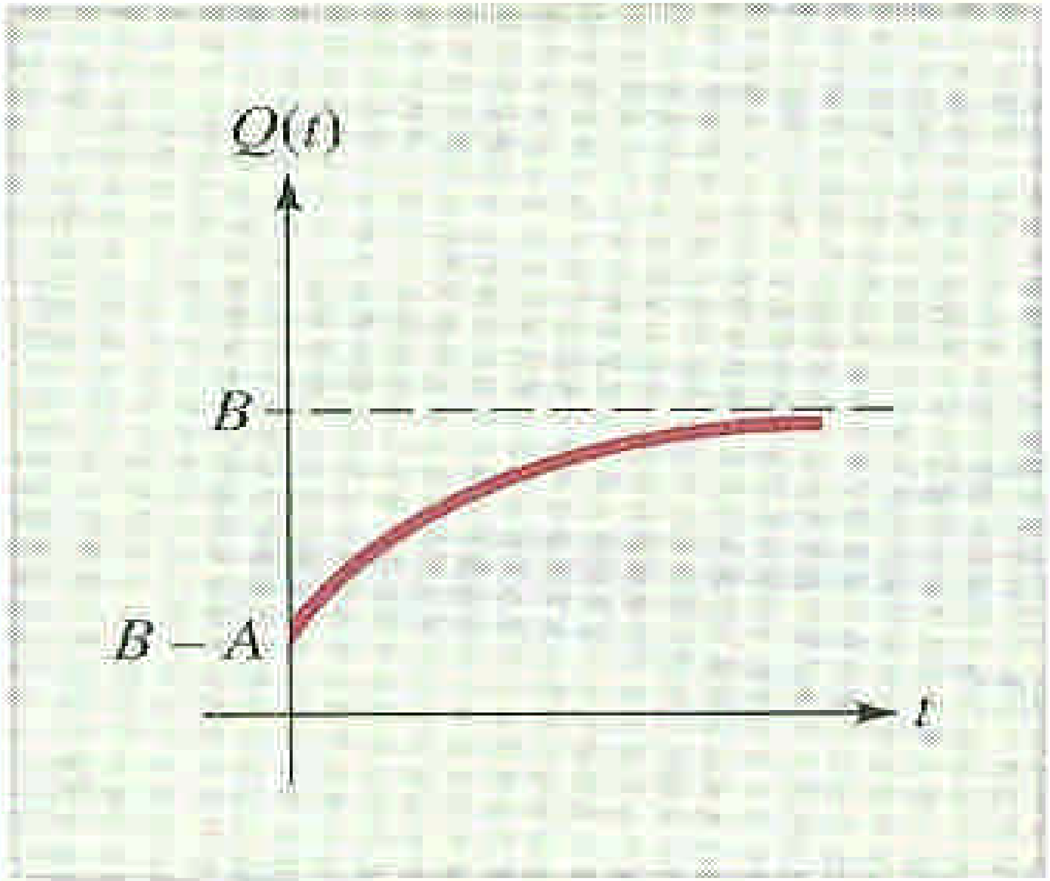

FIGURE 6.3 A learning curve: $Q(t) = B - Ae^{-kt}$.

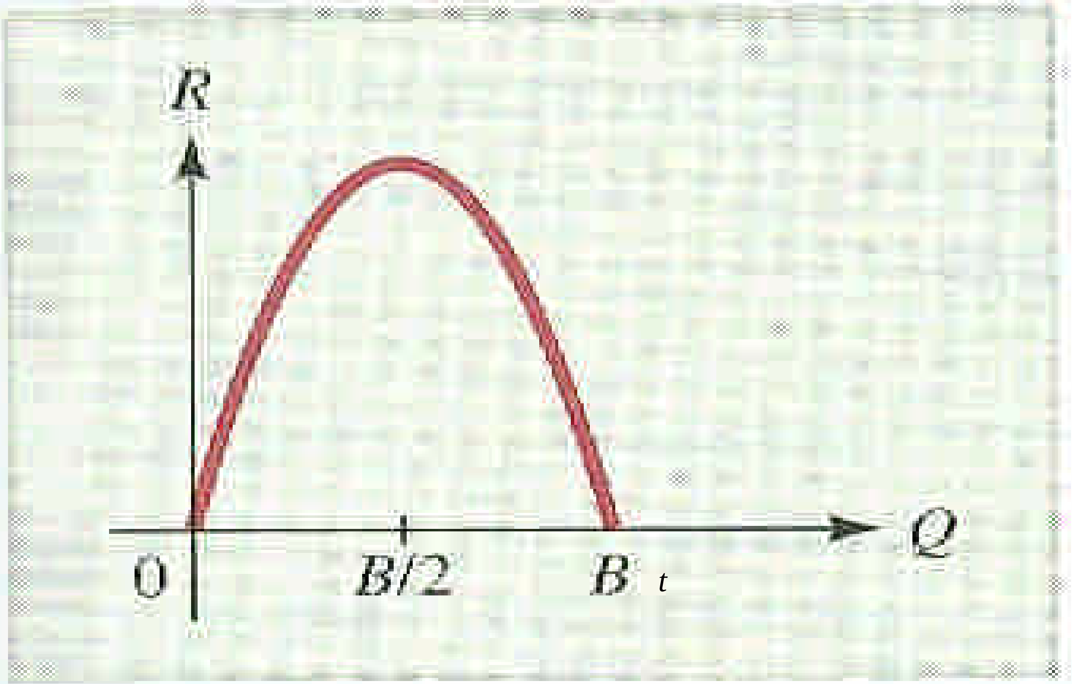

FIGURE 6.4 Graph of the population rate function $R(Q) = kQ(B - Q)$.

Example 6.2.8 (logistic growth)

• The rate at which an epidemic spreads through a community is jointly proportional to the number of residents who have been infected and the number of susceptible residents who have not.

• Q: Express the number of residents who have been infected as a function of time.

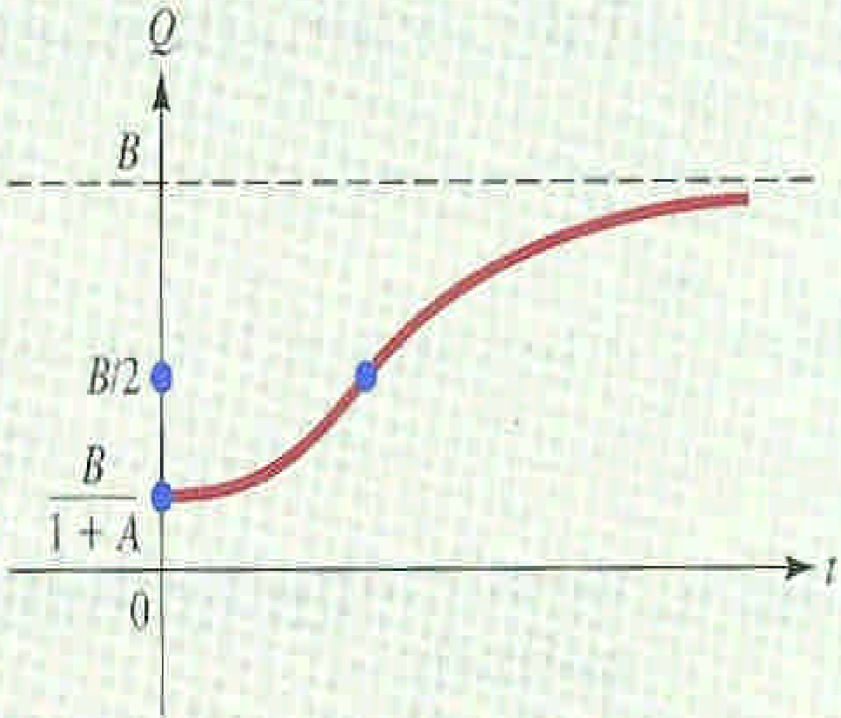

FIGURE 6.6 The logistic curve $y = \dfrac{B}{1 + Ae^{-Bkt}}$ showing the number of residents $Q(t)$ infected at time t during an epidemic.

Example 6.2.9 (dilution models)

- The residents of a certain community have voted to discontinue the fluoridation (加氟) of their water supply.
- The local reservoir currently holds 200 million gallons of fluoridated water that contains 1,600 pounds of fluoride.
- The fluoridated water is flowing out of the reservoir at the rate of 4 million gallons per day and is being replaced at the same rate by unfluoridated water.
- At all times, the remaining fluoride is evenly distributed in the reservoir.

- Q: Express the amount of fluoride in the reservoir as a function of time.

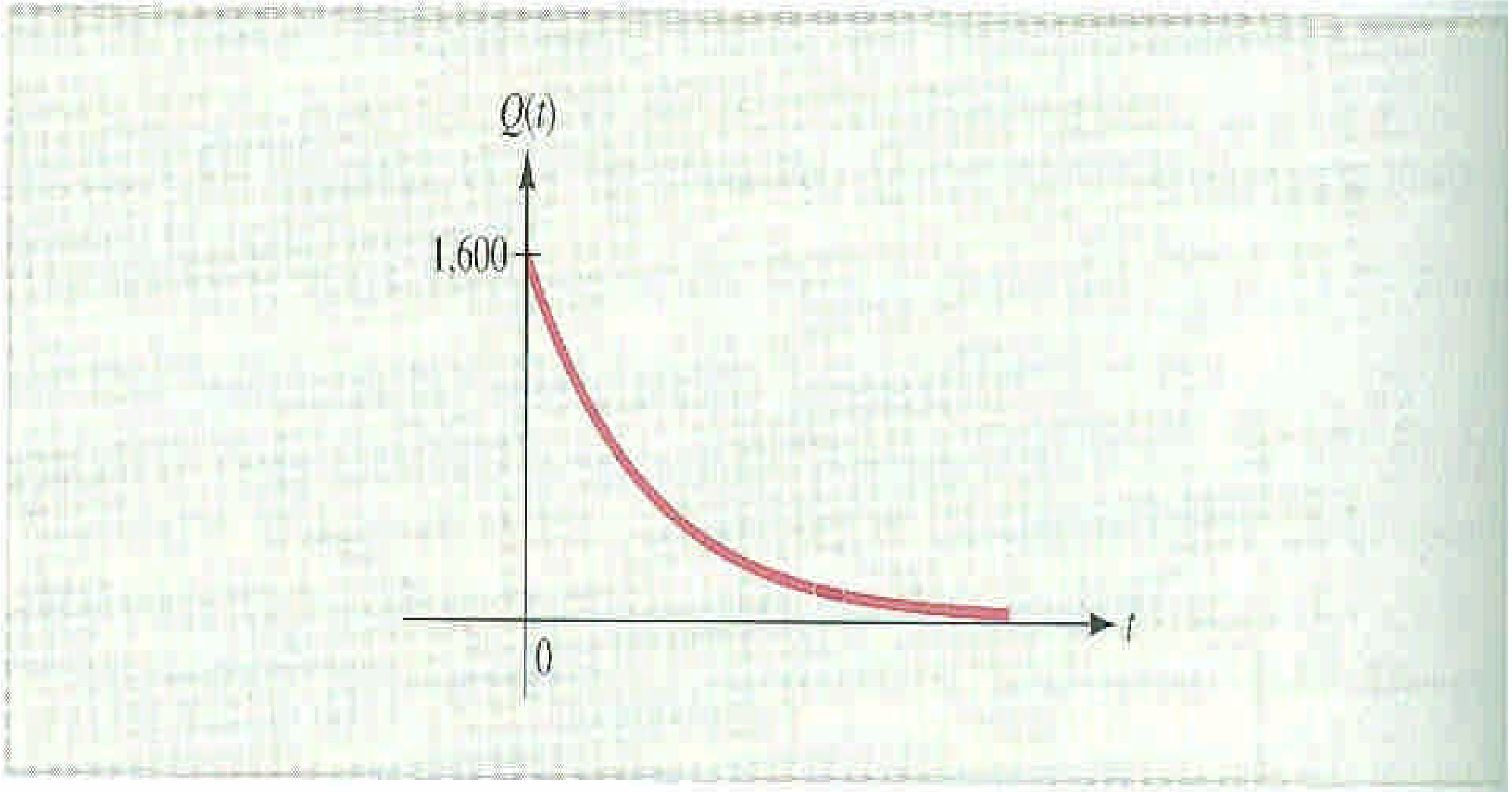

FIGURE 6.7 The amount of fluoride $Q(t) = 1{,}600e^{-t/50}$.

Example 6.2.10

Suppose the price $p(t)$ of a particular commodity varies in such a way that its rate of change with respect to time is proportional to the shortage $C - S$, where $D(p)$ and $S(p)$ are the linear demand and supply functions $D = 8 - 2p$ and $S = 2 + p$.

a.If the price is \$5 when t = 0 and \$3 when t = 2, find $p(t)$.
b.Determine what happens to $p(t)$ in the "long run" (as $t \to +\infty$).

SECTION 6.3
Improper Integrals (瑕積分); Continuous Probability

- **The Improper Integral**

If $f(x)$ is continuous for $x \geqq a$, then

$$\int_a^{-\infty} f(x)dx = \lim_{N \to +\infty} \int_a^N f(x)dx$$

If the limit exists, the improper integral is said to **converge** to the value of the limit.

If the limit does not exist, the improper integral **diverges**.

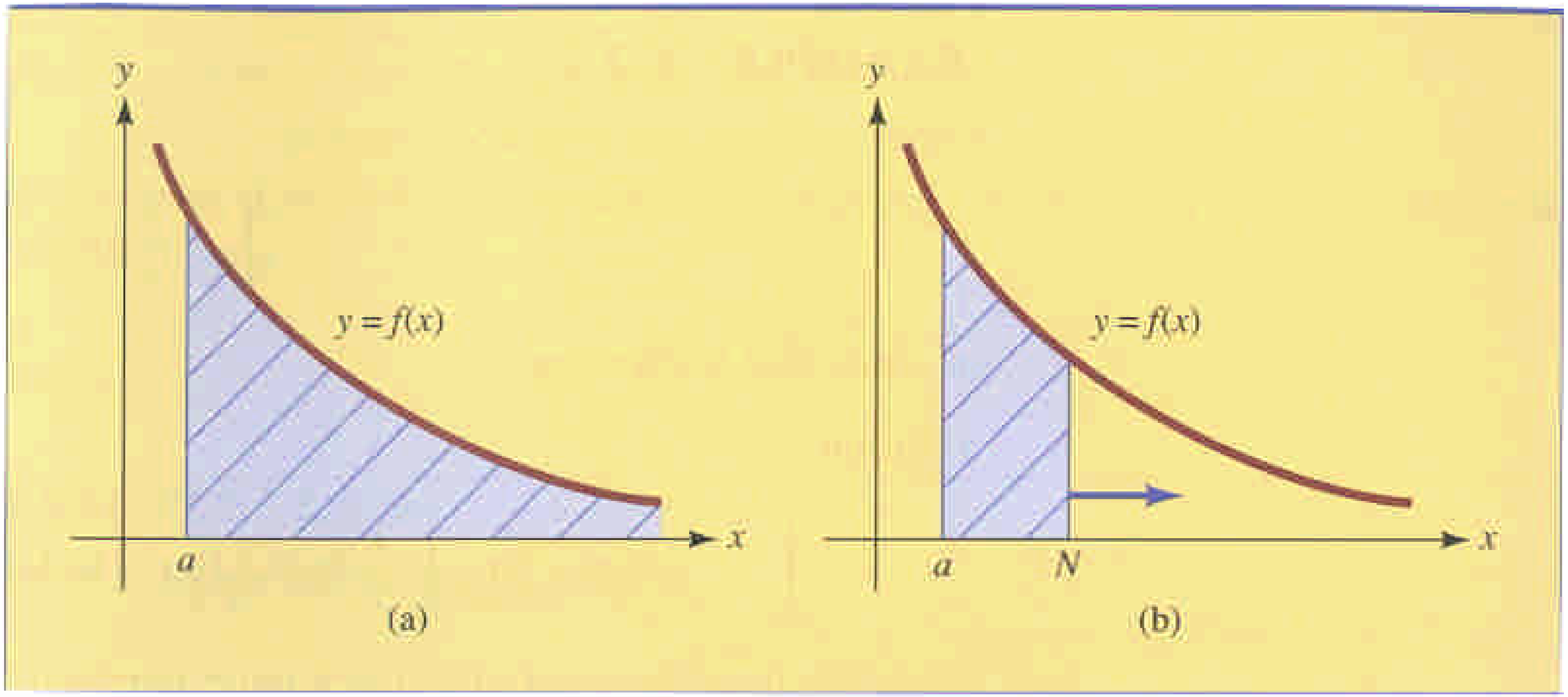

FIGURE 6.9 Area $= \int_a^{+\infty} f(x)\,dx = \lim_{N \to +\infty} \int_a^N f(x)\,dx.$

Example 6.3.1-3

Either evaluate the improper integral or show that it diverges.

$$\int_a^{+\infty} \frac{1}{x^2}\,dx$$

$$\int_a^{+\infty} \frac{1}{x}\,dx$$

$$\int_0^{+\infty} xe^{-2x}\,dx$$

(Hint: A useful limit on p511)

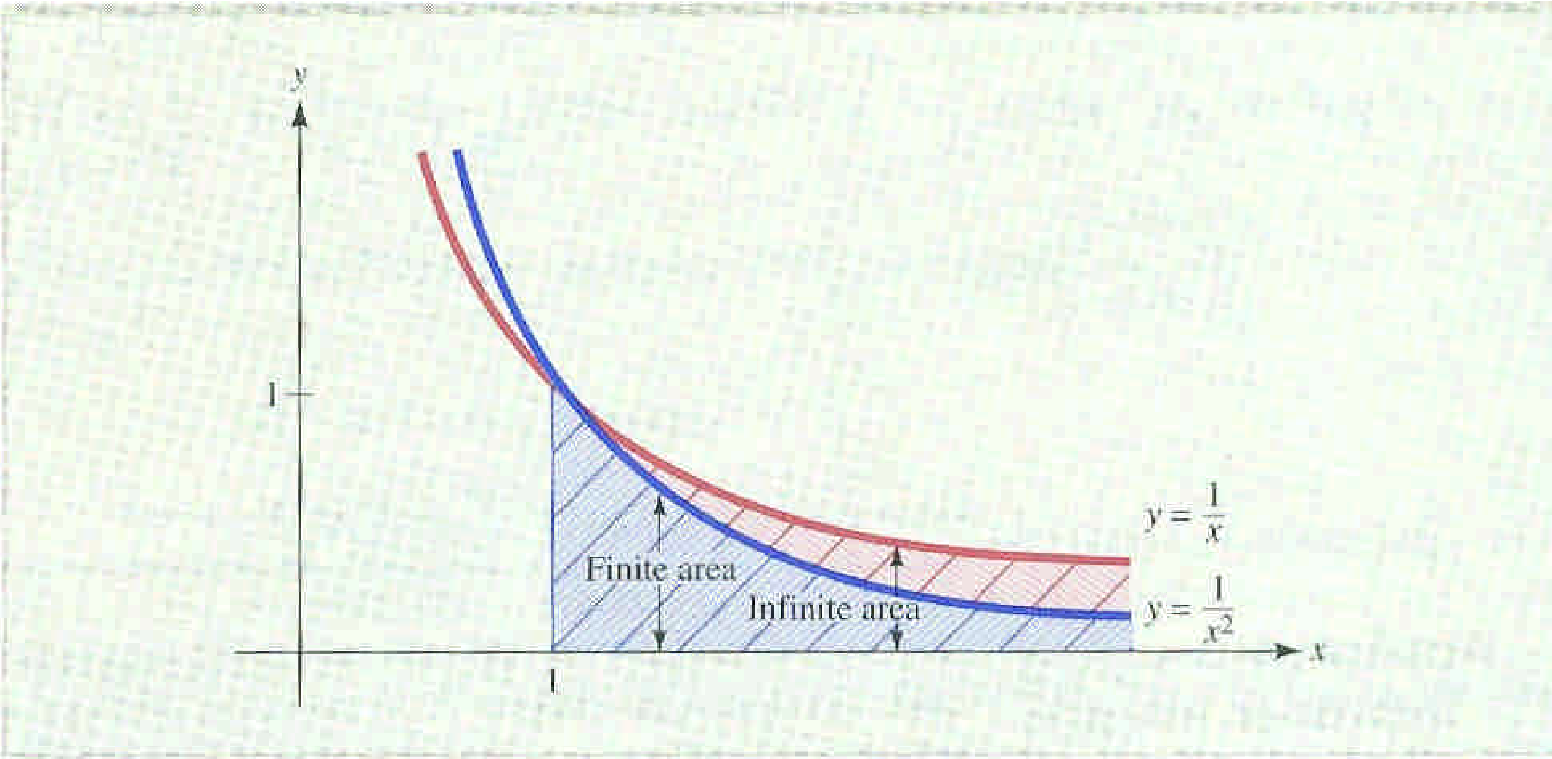

FIGURE 6.10 Comparison of the area under $y = \frac{1}{x}$ with that under $y = \frac{1}{x^2}$.

Example 6.3.4

・A donor wishes to endow a scholarship at a local college with a gift that provides a continuous income stream at the rate of 25,000+1,200t dollars per year in perpetuity (永久).

・Assuming the prevailing annual interest rate stays fixed at 5% compounded continuously, what donation is required to finance the endowment?

Example 6.3.5

- It is estimated that t years from now, a certain nuclear power plant will be producing radioactive waste at the rate of $f(t)=400t$ pounds per year.

- The waste decays exponentially at the rate of 2% per year.

- Q: What will happen to the accumulation of radioactive waste from the plant in the long run?

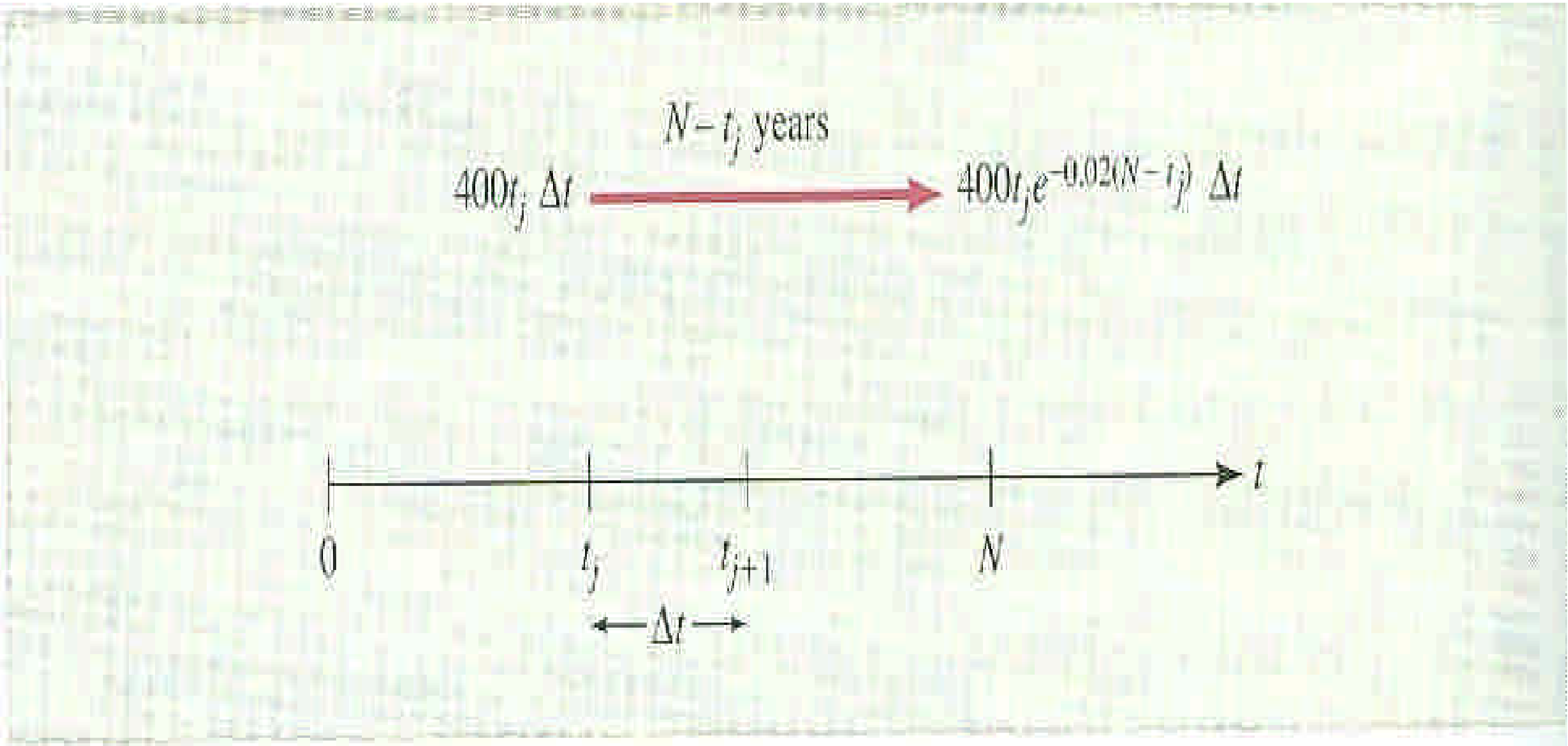

FIGURE 6.11 Radioactive waste generated during the jth subinterval.

Continuous probability function

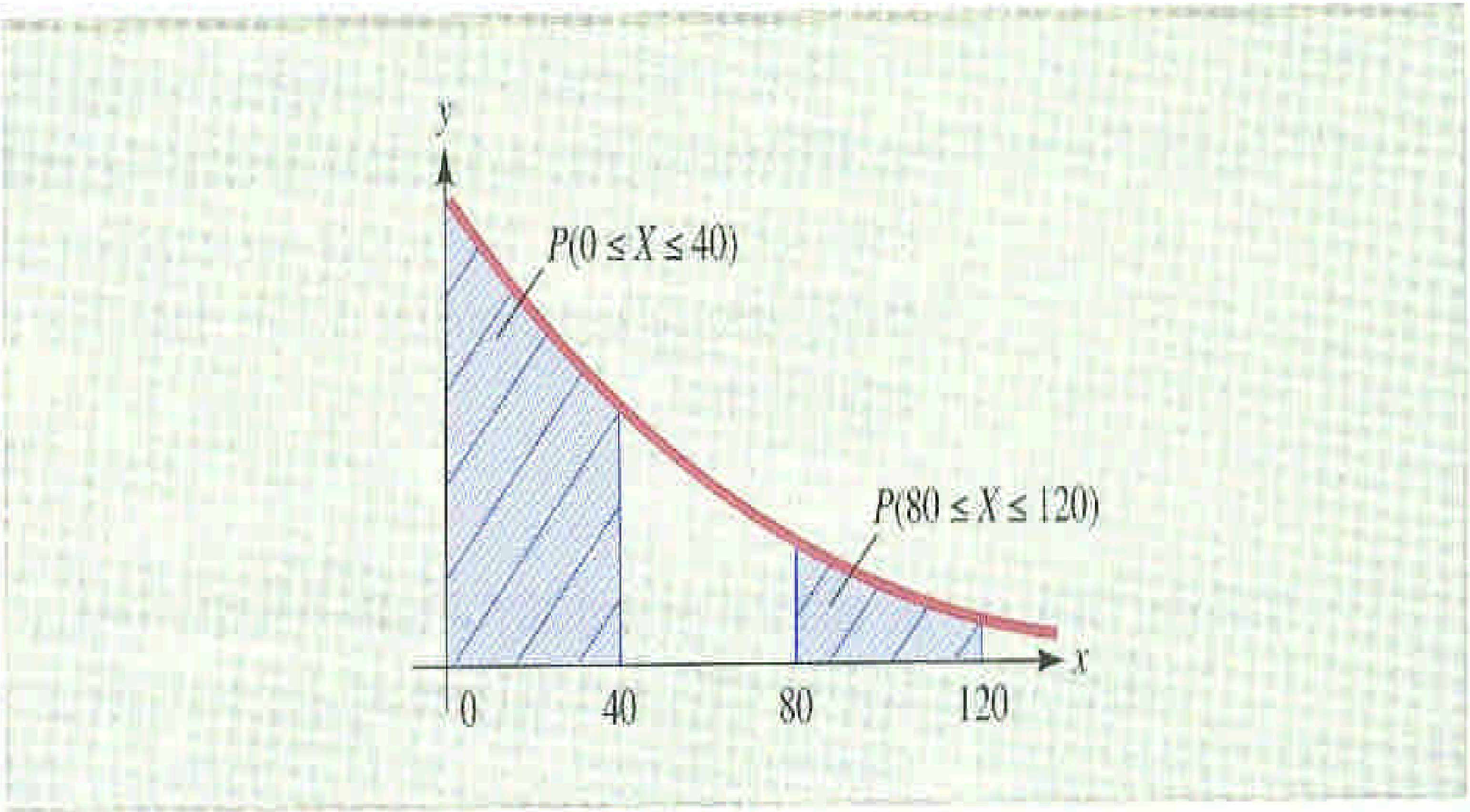

FIGURE 6.12 A possible probability density function for the life span of a lightbulb.

Continuous Probability

Probability Density Functions (PDF)

A probability density function for the continuous random variable X is a function $f(x)$ that satisfies the following three conditions:

1. $f(x) \ge 0$ for all real x
2. The total area under the graph of $f(x)$ is 1
3. The probability that X lies in the interval $a \le X \le b$ is given by the integral

$$P\left(a \le X \le b\right) = \int_a^b f(x)\,dx$$

4. Consider the improper integrals,

P($-\infty < X < \infty$) and P($X \geqq a$).

A uniform pdf (see Figure 6.13)

- A uniform pdf is constant over a bounded interval $A \leqq X \leqq B$ and zero outside that interval.
- A random variable that has a uniform density function is said to be uniformly distributed.
- Roughly speaking, for a uniformly distributed random variable, all values in some bounded interval are "equally likely".
- Example: the waiting time of a motorist at a traffic light that remains red for, say, 40 seconds at a time. This random variable has a uniform distribution because all waiting times between 0 and 40 seconds are equally likely.

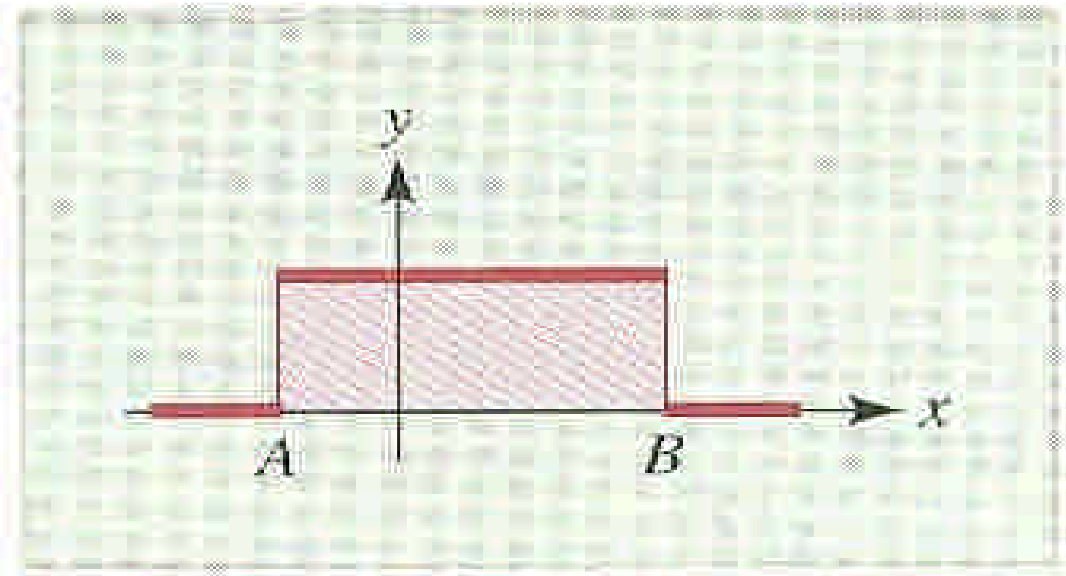

FIGURE 6.13 A uniform density function.

○Uniform Density Function (p517)

$$f(x) = \begin{cases} \dfrac{1}{B - A} & \text{if} \quad A \le x \le B \\ 0 & \text{otherwise} \end{cases}$$

Example 6.3.6

- A certain traffic light remains red for 40 seconds at a time.

- You arrive (at random) at the light and find it red.

- Use an appropriate uniform density function to find the probability that you will have to wait at least 15 seconds for the light to turn green.

- **Exponential Density Function**

$$f(x) = \begin{cases} ke^{-kx} & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

A random variable that has an exponential density function is said to be **exponentially distributed.**

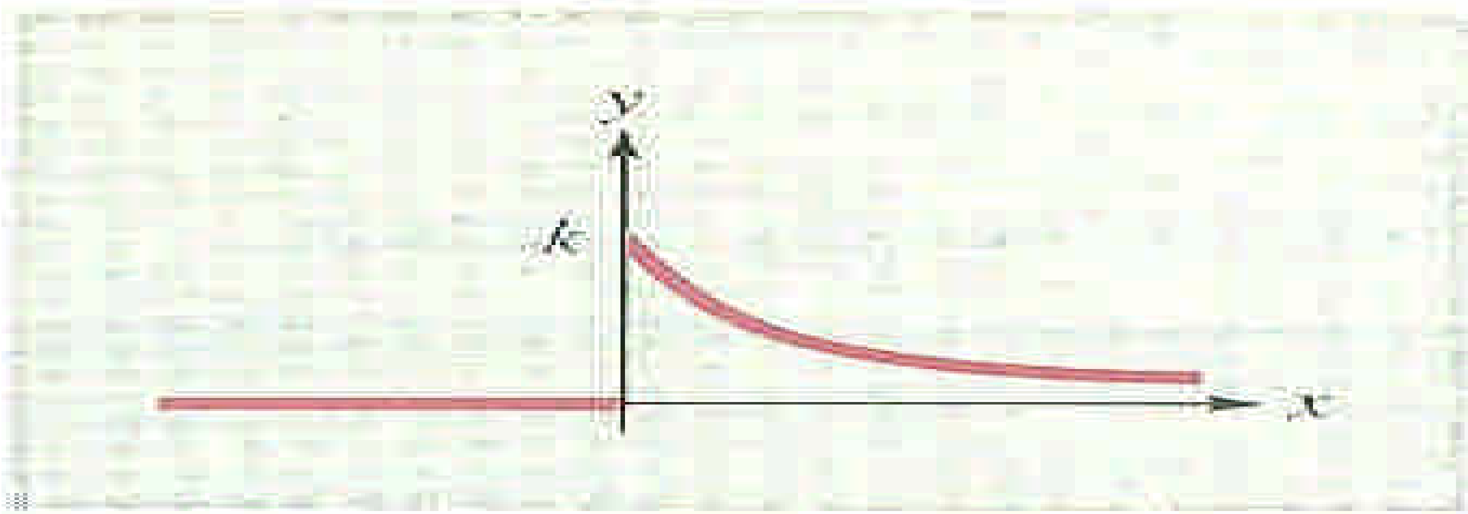

FIGURE 6.14 An exponential density function.

Example 6.3.7

- Let X be a random variable that measures the duration of telephone calls in a certain city and suppose that a probability density function for X is

$$f(x) = \begin{cases} 0.5e^{-0.5x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

where x denotes the duration (in minutes) of a randomly selected call.

Q1: Find the probability that a randomly selected call will last between 2 and 3 minutes.

Q2: Find the probability that a randomly selected call will last at least 2 minutes.

The most widely used probability density functions are the **normal density functions**.

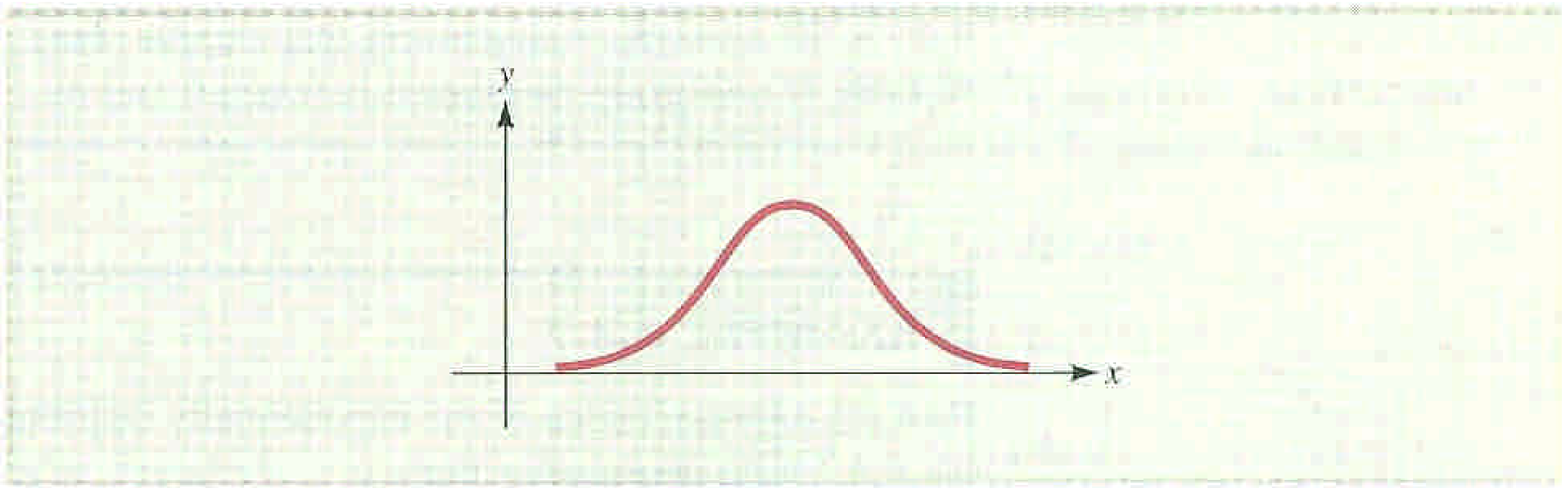

FIGURE 6.15 Graph of a normal density function.

Expected Value

If X is a continuous random variable with probability density function f, the expected value (or mean) of X is

$$E(x) = \int_{-\infty}^{+\infty} x f(x)\,dx$$

Example 6.3.8

Find the expected value of the uniformly distributed random variable from Example 6.3.6 with density function

$$f\left(x\right)=\begin{cases}\dfrac{1}{40} & \text{if } \leq x \leq 40\\ 0 & \text{otherwise}\end{cases}$$

Example 6.3.9

Find the expected value of the exponentially distributed random variable from Example 6.3.7 with density function

$$f(x) = \begin{cases} 0.5e^{-0.5x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

SECTION 6.4
Numerical Integration

- Numerical methods are needed when the function to be integrated does not have an elementary antiderivative.

Approximation by rectangles

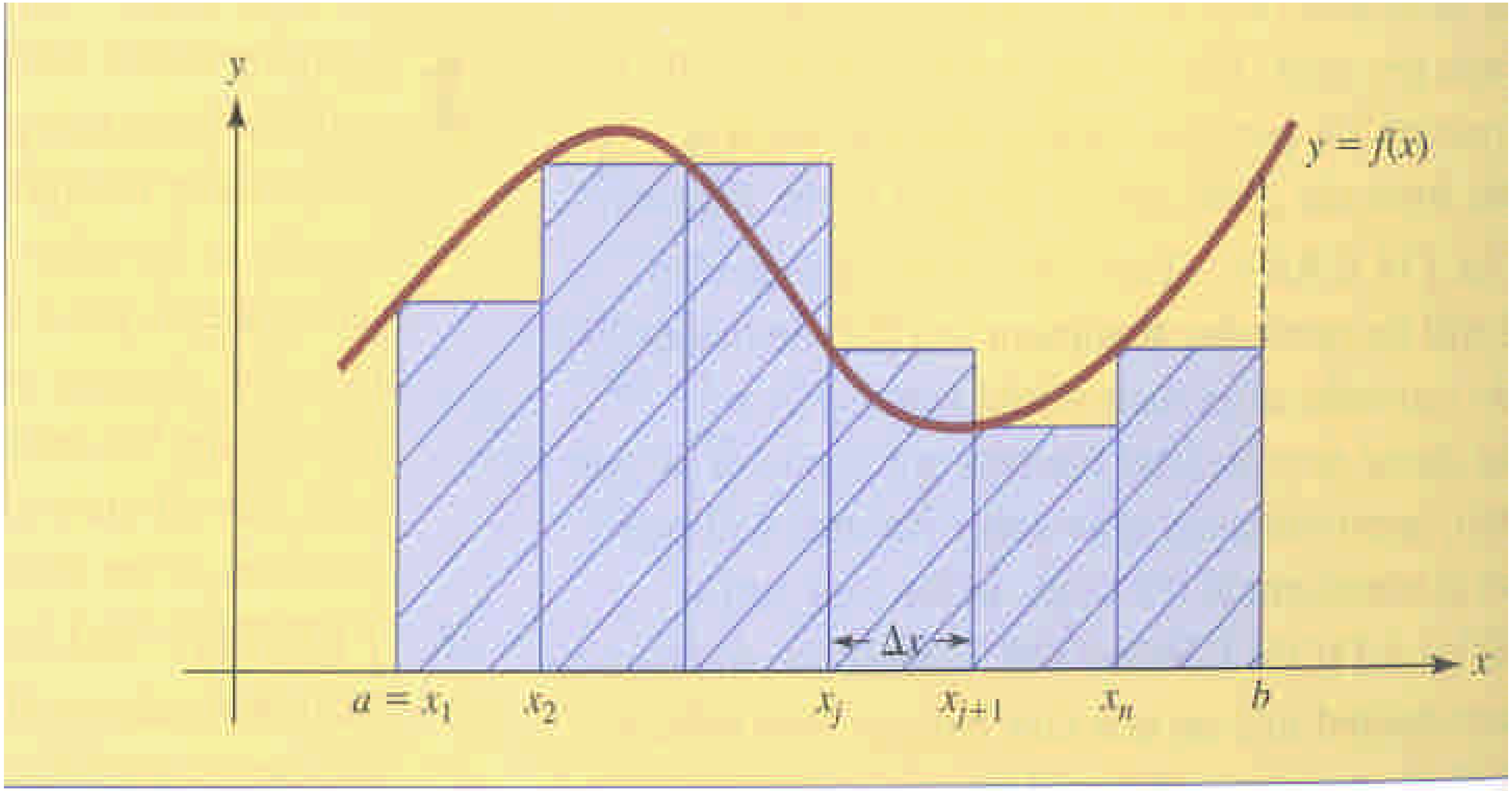

FIGURE 6.16 Approximation by rectangles.

The Trapezoidal Rule (see p527)

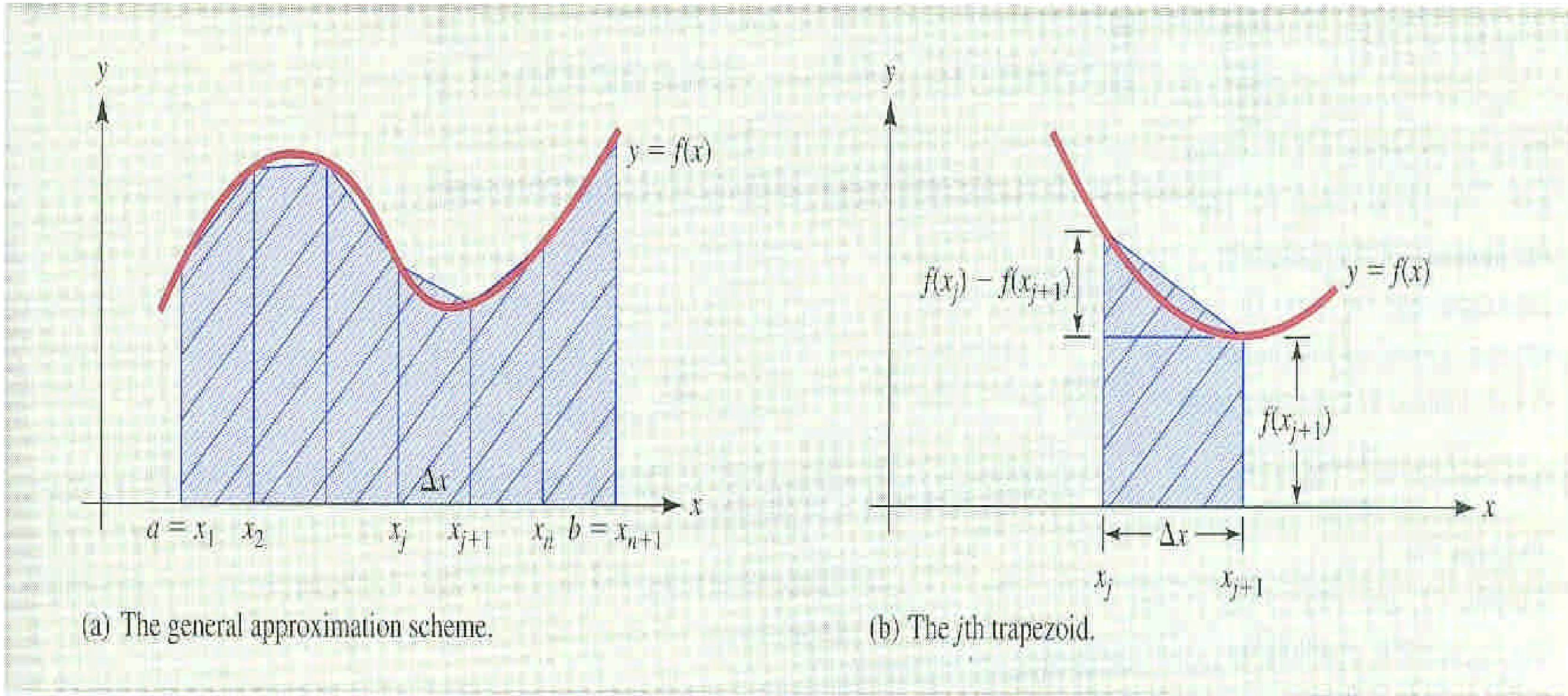

FIGURE 6.17 Approximation by trapezoids.

Note: The accuracy of the approximation improves significantly if trapezoids are used instead of rectangles.

Example 6.4.1

Use the trapezoidal rule with n=10 to approximate

$$\int_1^2 \frac{1}{x}\,dx$$

Note: Error estimate for the trapezoidal rule (p528).

Example 6.4.2

Estimate the accuracy of the approximation of

$$\int_1^2 \frac{1}{x}\, dx$$

by the trapezoidal rule with n=10.

Ans: 0.00167

Example 6.4.3

How many subintervals are required to guarantee that the error will be less than 0.00005 in the approximation of

$$\int_1^2 \frac{1}{x}\,dx$$

using the trapezoidal rule?

Ans: 58 subintervals are required for achieving the above accuracy (**too many subintervals**).

Approximation Using Parabolas: Simpson's Rule (p530)

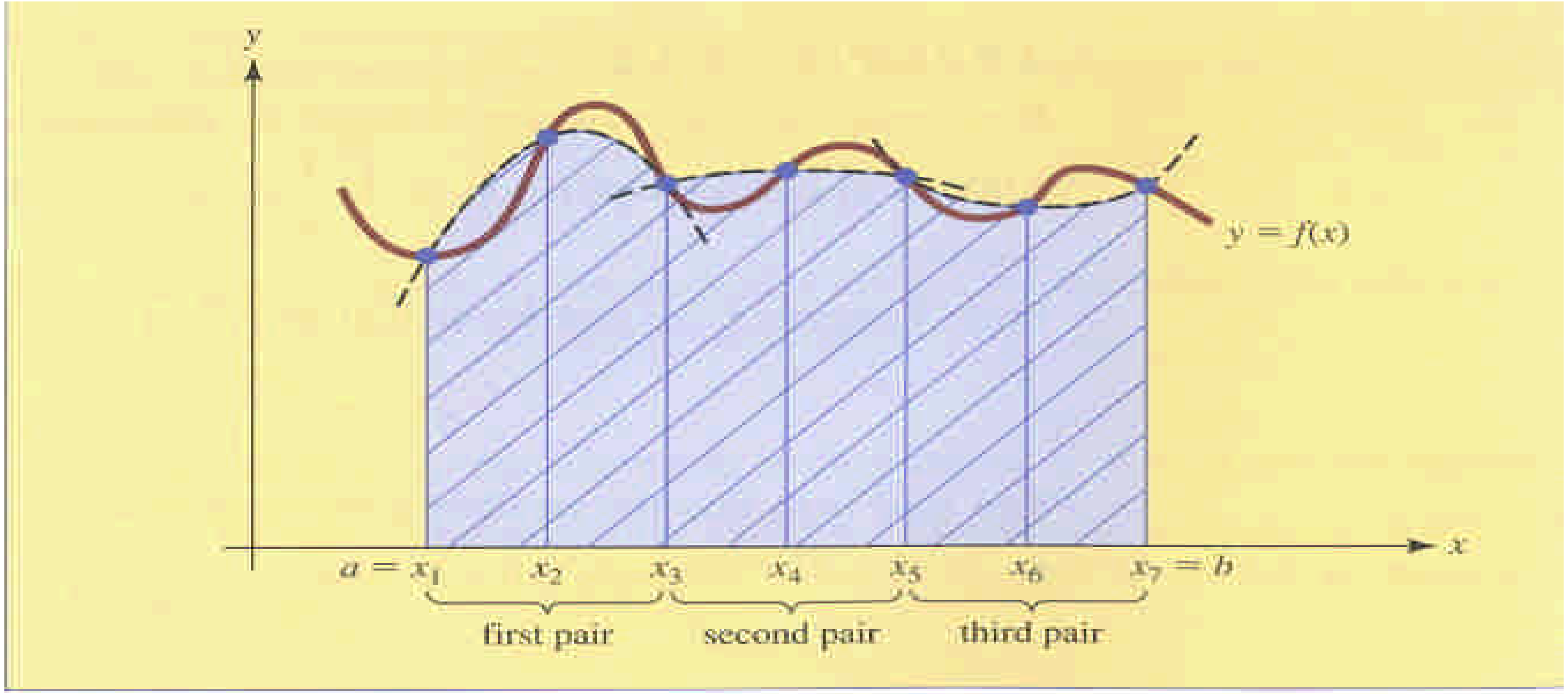

FIGURE 6.19 Approximation using parabolas.

Note: Comparing to Trapezoidal rule, Simpson's rule requires substantially fewer calculations to achieve a specified degree of accuracy.

Example 6.4.4

Use Simpson's rule with n=10 to approximate

$$\int_1^2 \frac{1}{x}\, dx$$

Note: Error estimate for Simpson's rule (p531)

Example 6.4.5

Estimate the accuracy of the approximation of

$$\int_1^2 \frac{1}{x}\,dx$$

by Simpson's rule with n=10.

Ans: 0.000013

Example 6.4.6

How many subintervals are required to ensure accuracy to within 0.00005 in the approximation of $\int_1^2 \frac{1}{x}\,dx$

by Simpson's rule?

Ans: 8 subintervals are needed to achieve the above accuracy.

Notes

- All methods use the area under the shapes of curves selected to approximate the area under the true curve.
- Timing for determining the approximation methods?
- Error estimation is a criterion for choosing the approximation method, given the same number of subintervals.

Example 6.4.7

• Jack needs to know the area of his swimming pool in order to buy a pool cover, but this is difficult because of the pool's irregular shape.

• Suppose Jack makes the measurements shown in Figure 6.20 at 4-ft intervals along the base of the pool (all measurements are in feet).

•Q: How can he use the trapezoidal rule to estimate the area?

Example 6.4.7

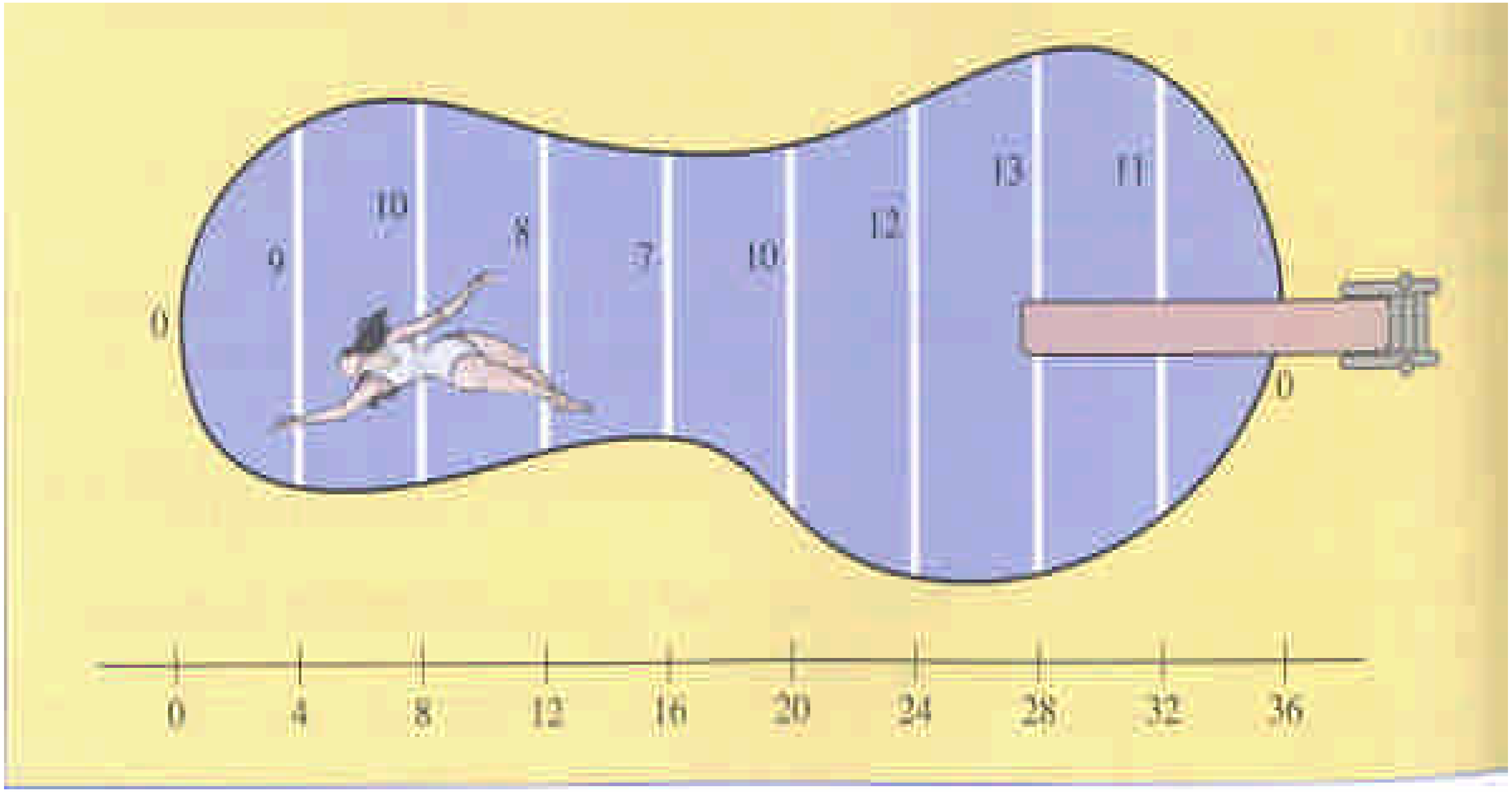

FIGURE 6.20 Measurements across a pool.