CHAPTER 7
Calculus of Several Variables

- Function of Several Variables
- Partial Derivatives
- Optimizing Functions of Two Variables
- The Method of Least-Squares
- Constrained Optimization: The Method of Lagrange Multipliers
- Double Integrals

SECTION 7.1
Functions of Several Variables

- **Function of Two Variables**

  A function f of the two independent variables x and y is a rule that assigns to each ordered pair (x, y) in a given set D (the domain of f) exactly one real number, denoted by f(x, y).

Q: Why do we consider the functions of more than one independent variable? See examples on p558.

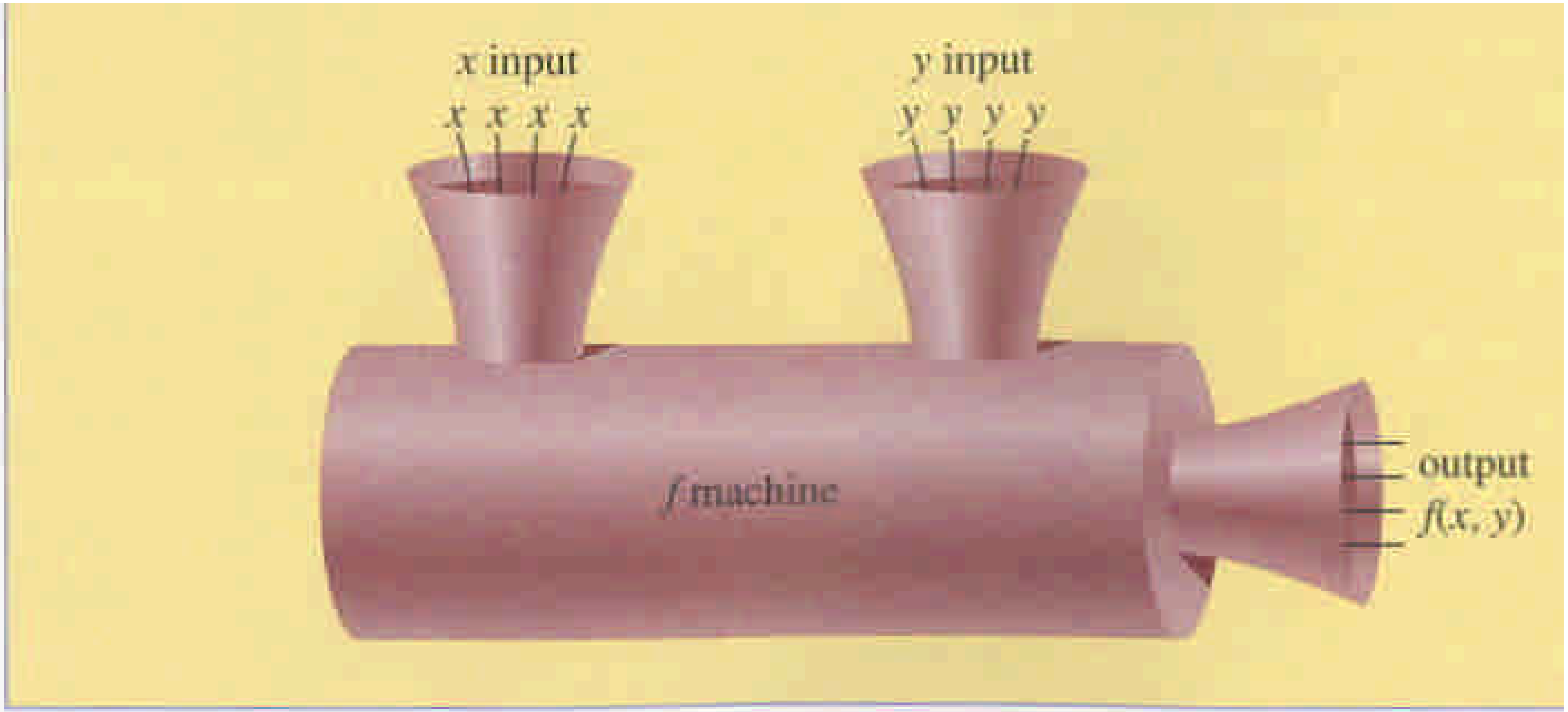


FIGURE 7.1 A function of two variables as a "machine."

Example 7.1.1

Suppose $f(x, y) = \dfrac{3x^2 + 5y}{x - y}$

a. Find the domain of *f*.

a. Compute *f(1, -2)*.

Example 7.1.2

Suppose $f(x, y) = xe^y + \ln x.$

a. Find the domain of *f*.

a. Compute $f(e^2, \ln x)$

Example 7.1.3

Given the function of three variables $f(x,y,z)=xy+xz+yz$, evaluate $f(-1,2,5)$.

Example 7.1.4

- A sports store in St. Louis carries two kinds of tennis rackets,
  - the Serena Williams autograph (親筆) brands and
  - the Jennifer Capriati autograph brands.
- The consumer demand for each brand depends
  - not only on its own price,
  - but also on the price of the competing brand.

Example 7.1.4

- Sales figures indicate that if the Williams brand sells for x dollars per racket and the Capriati brand for y dollars per racket,
  - the demand for Williams rackets will be $D_1=300-20x+30y$ rackets per year and
  - the demand for Capriati rackets will be $D_2=200+40x-10y$ rackets per year.

- **Q**: Express the store's total annual revenue from the sale of these rackets as a function of the prices x and y.

Example 7.1.5

- Suppose that at a certain factory, output is given by the Cobb-Douglas production function $Q(K,L)=60K^{1/3}L^{2/3}$ units, where
  - K is the capital investment measured in units of \$1,000 and
  - L the size of the labor force measured in worker-hours.

Q1: Compute the output if the capital investment is \$512,000 and 1,000 worker-hours of labor are used.

Q2: Show that the output in part (a) will double if both the capital investment and the size of the labor force are doubled.

Example 7.1.6

Recall (from Section 4.1) that the present value of B dollars in t years invested at the annual rate r compounded k times per ear is given by

$$P(B,r,k,t) = B\left(1+\frac{r}{k}\right)^{-kt}$$

Find the present value of $10,000 in 5 years invested at 6% per ear compounded quarterly.

Example 7.1.7

A population that grows exponentially satisfies

$$P(A,k,t) = Ae^{kt}$$

where P is the population at time t, A is the initial population (when $t=0$), and k is the relative (per capita) growth rate.

The population of a certain country is currently 5 million people and is growing at the rate of 3% per year.

What will the population be in 7 years?

Graphs of functions of two variables

- The graph of a function of two variables $f(x,y)$ is the set of all triples (x,y,z) such that (x,y) is in the domain of f and $z=f(x,y)$.
- A three-dimensional coordinate system, see Figure 7.2 (p562).
- Figure 7.3, the graph of $z=f(x,y)$
- Figure 7.4: Several surfaces in 3-dimensional space.
- Level curves (see p562 and Figure 7.5)

Graphs of functions of two variables

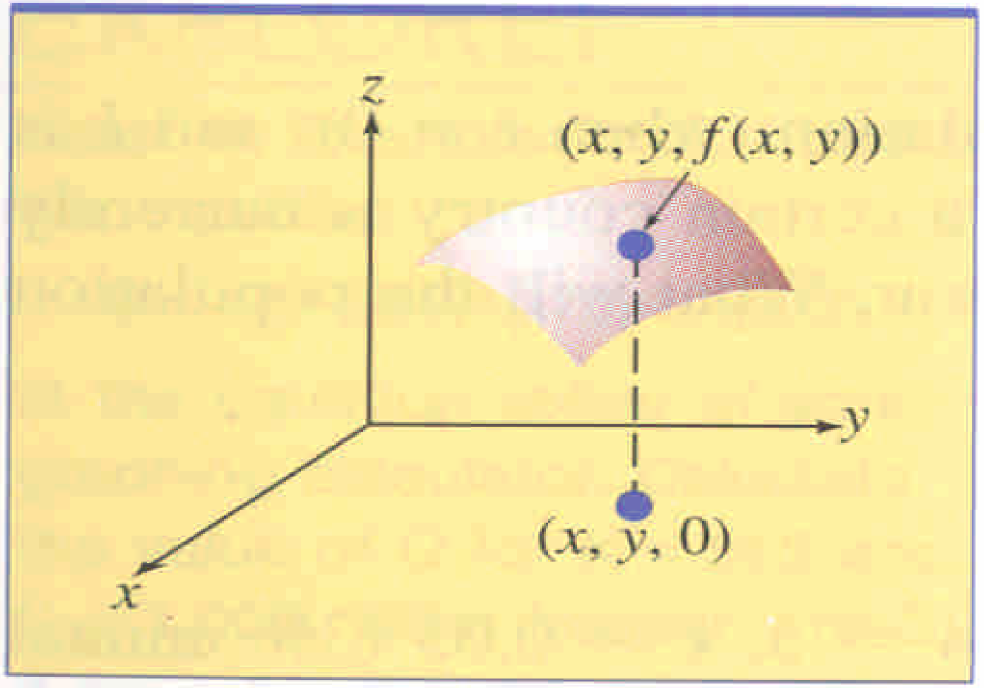
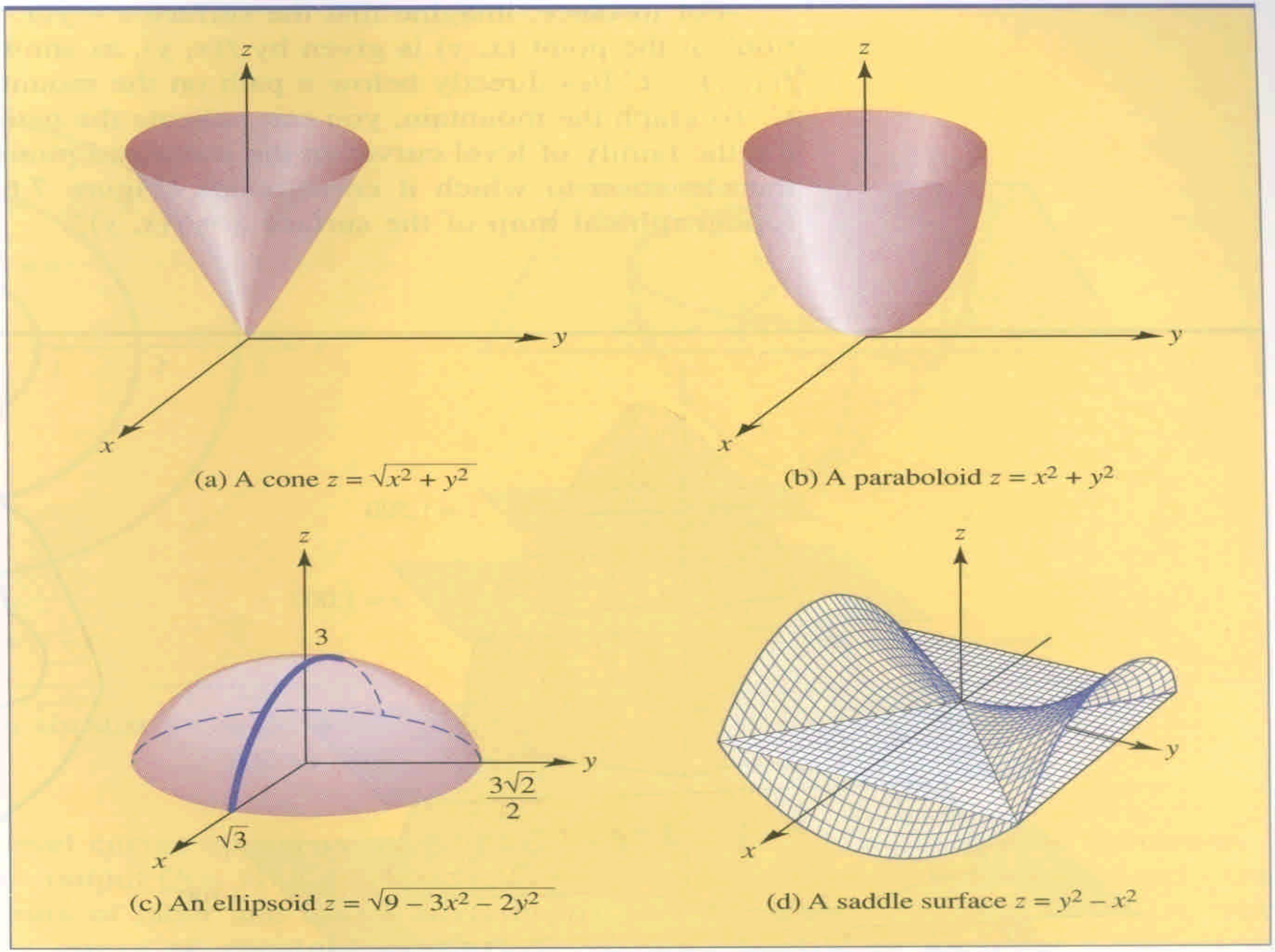


FIGURE 7.3 The graph of $z = f(x, y)$.



(a) A cone $z = \sqrt{x^2 + y^2}$

(b) A paraboloid $z = x^2 + y^2$

(c) An ellipsoid $z = \sqrt{9 - 3x^2 - 2y^2}$

(d) A saddle surface $z = y^2 - x^2$

FIGURE 7.4 Several surfaces in three-dimensional space.

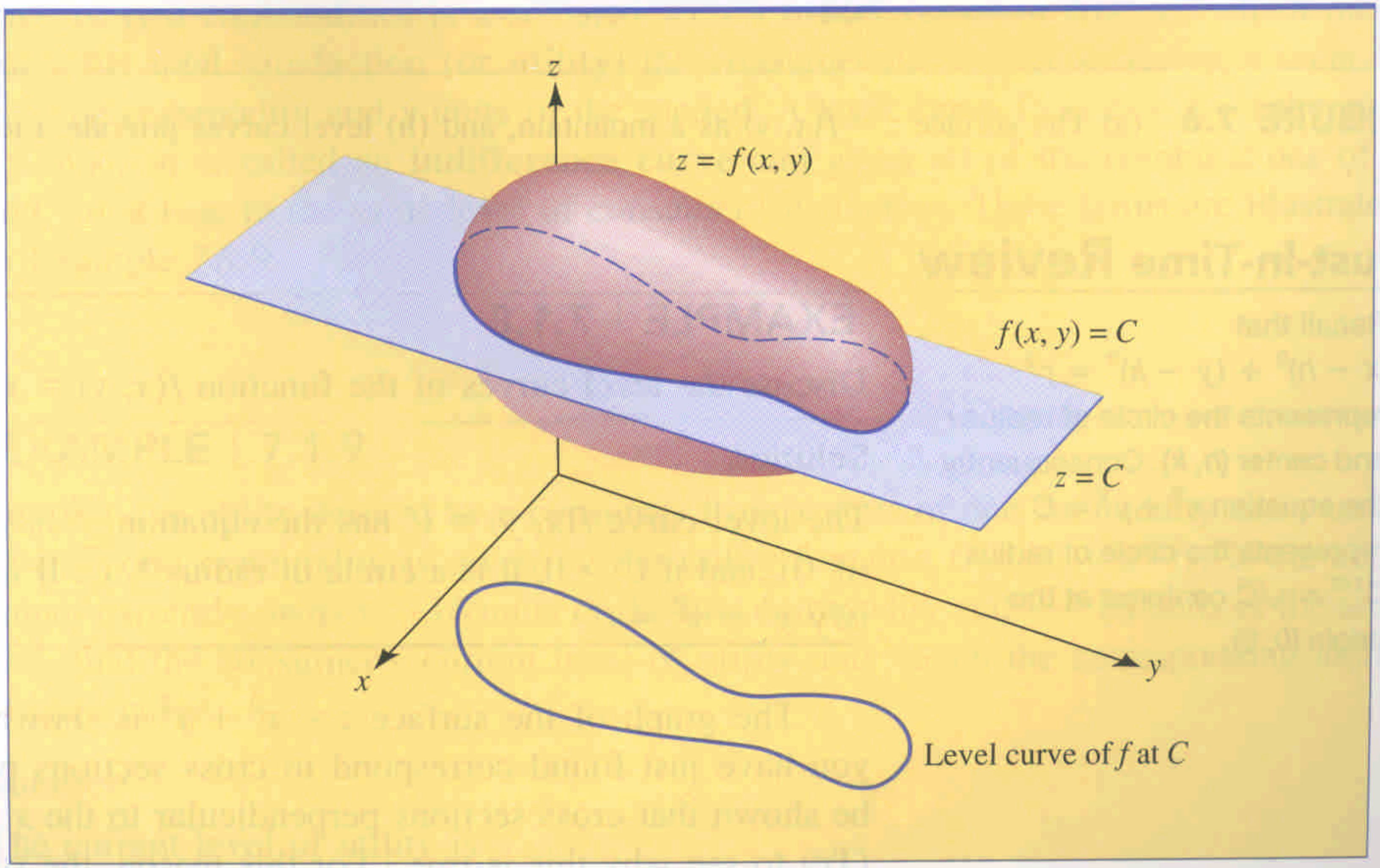


FIGURE 7.5 A level curve of the surface $z = f(x, y)$.

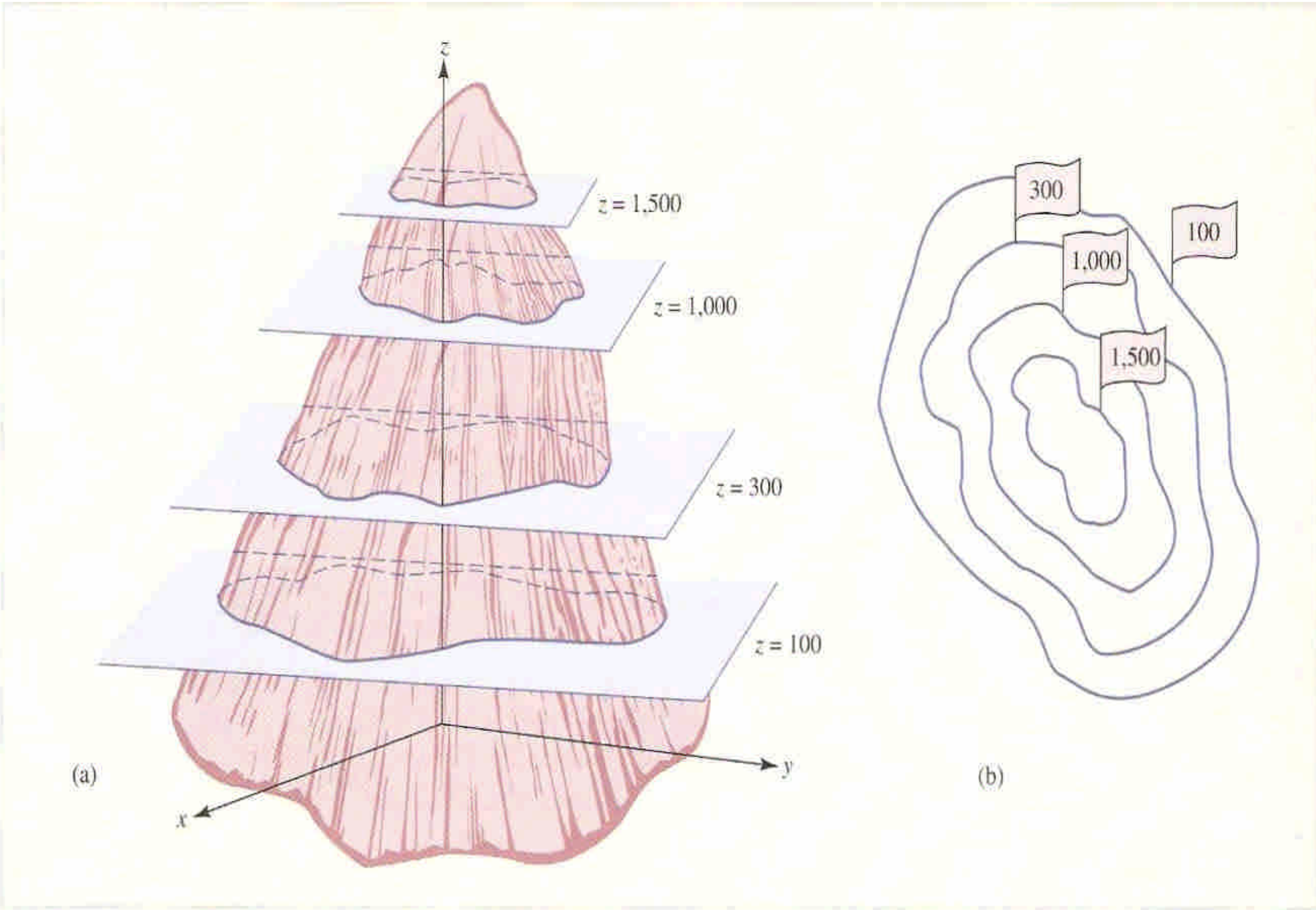


FIGURE 7.6 (a) The surface $z = f(x, y)$ as a mountain, and (b) level curves provide a topographical map of $z = f(x, y)$.

Example 7.1.8

Discuss the level curves of the function

$$f(x,y) = x^2 + y^2.$$

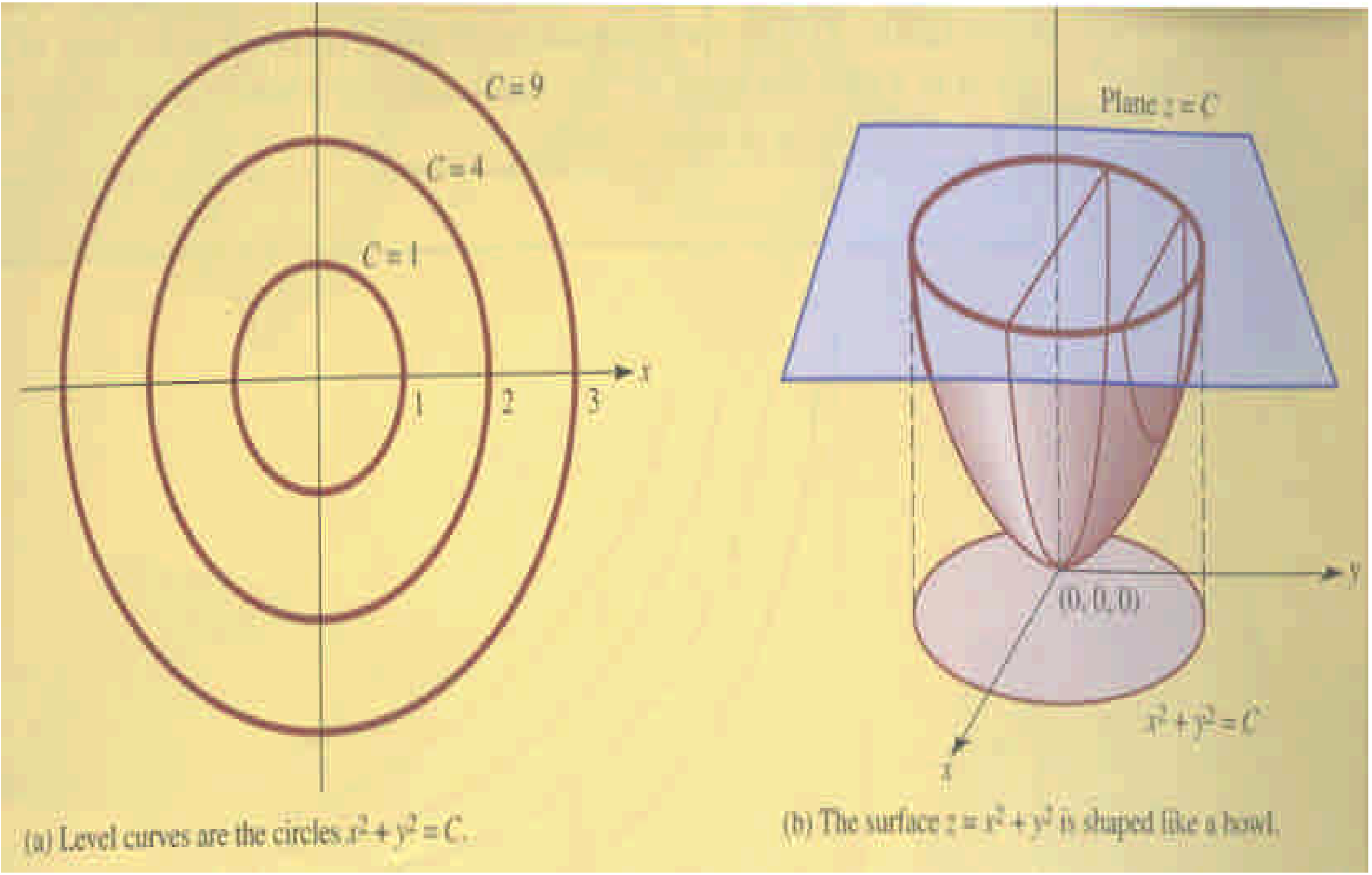


FIGURE 7.7 Level curves help to visualize the shape of surface.

Example 7.1.9

- Suppose the utility derived by a consumer from x units of one commodity and y units of a second commodity is given by the utility function $U(x,y)=x^{3/2}y$.

- If the consumer currently owns $x=16$ units of the first commodity and $y=20$ units of the second, find the consumer's current level of utility and sketch the corresponding indifference curve.

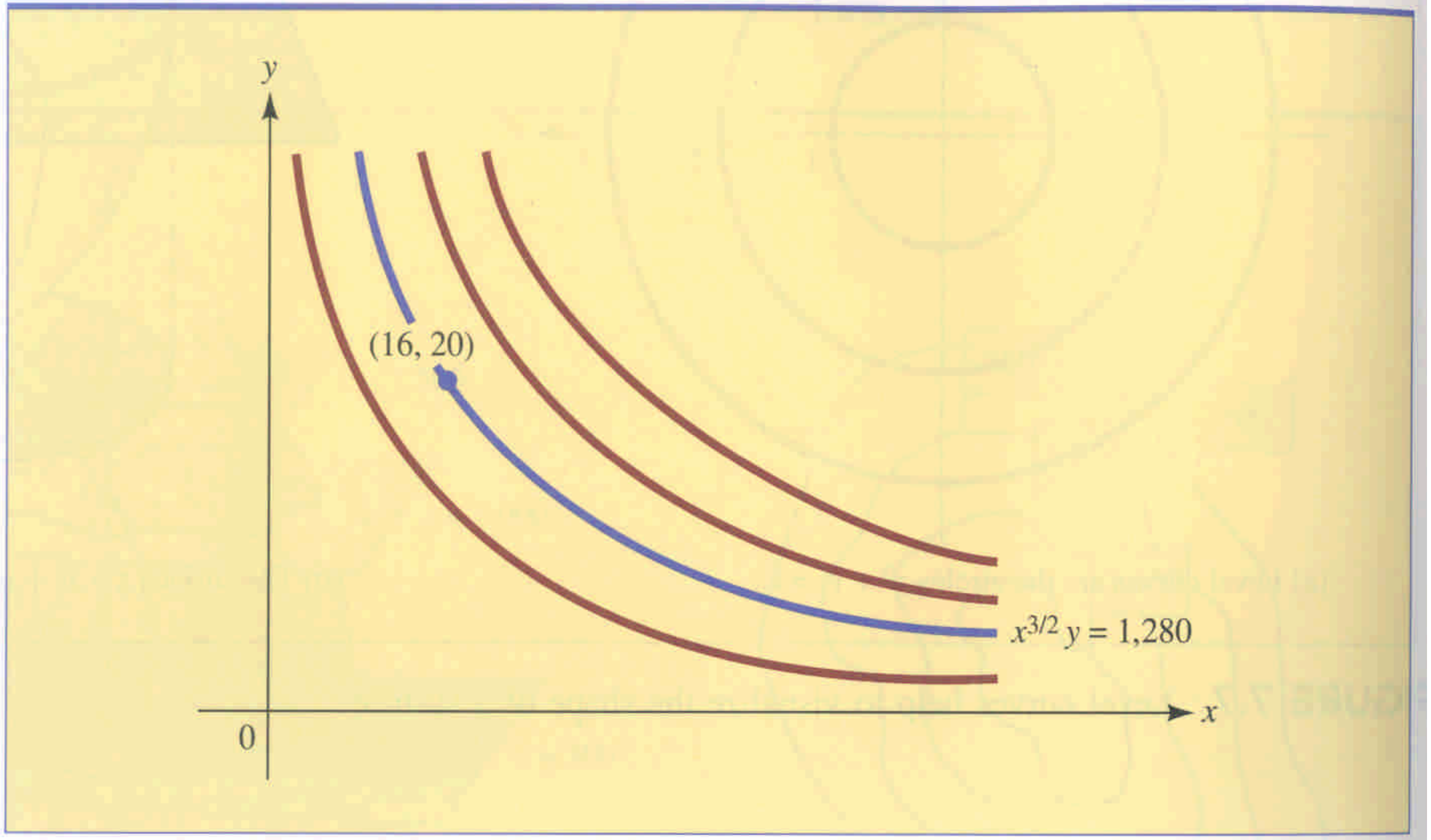


FIGURE 7.8 Indifference curves for the utility function $U(x, y) = x^{3/2}y$.