SECTION 7.3
Optimizing Functions of Two Variables

- **Relative Extrema**

The function $f(x,y)$ is said to have a **relative maximum**, f(a,b), at the point $P(a,b)$ in the domain of f if $f(a,b) \geq f(x,y)$ for all points (x,y) in a circular disk centered at P.

Similarly, if $f(c,d) \leq f(x,y)$ for all points (x,y) in a circular disk centered at Q, then $f(x,y)$ has a **relative minimum**, f(c,d), at $Q(c,d)$. See Figure 7.11.

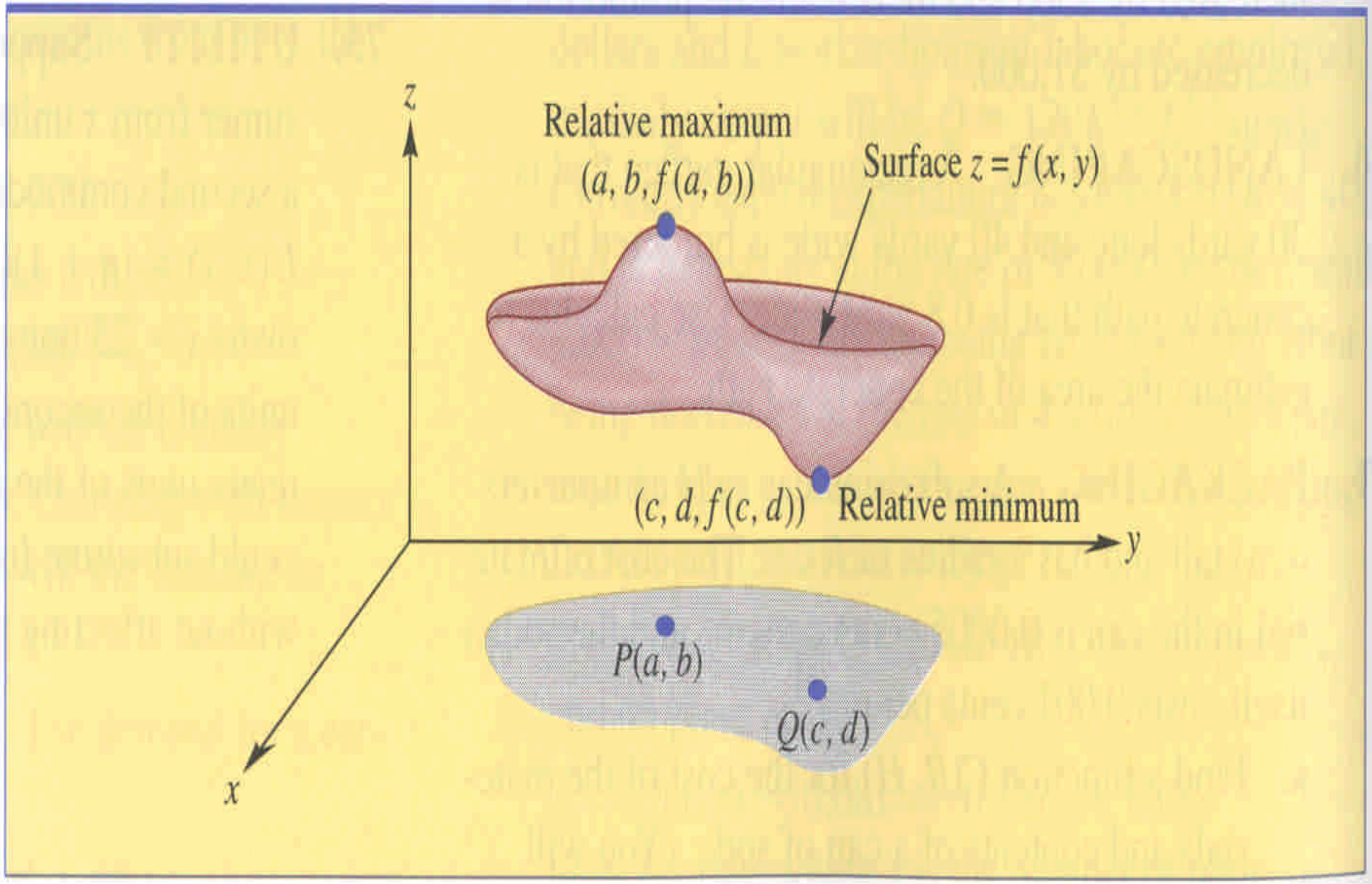

FIGURE 7.11 Relative extrema of the function $f(x, y)$.

Critical points

- Recall the critical numbers for functions of one variable.
- Similar the critical points are considered for the functions of more than one variable.
- The points (a,b) in the domain of f(x,y) for which both $f_x(a,b)=0$ and $f_y(a,b)=0$ are said to be critical points of f.
- These critical points play an important role in the study of relative maxima and minima.

Connection between critical points and relative extrema

- Suppose f(x,y) has a relative maximum at the point (a,b).
  - The curve formed by intersecting the surface z=f(x,y) with the vertical plane y=b has a relative maximum and hence a horizontal tangent line when x=a. Same for the vertical plane x=a.
  - The partial derivatives $f_x(a,b)=0$ and $f_y(a,b)=0$, see Figure 7.12.
- Fact: A point at which a function of two variables has a relative maximum (minimum) must be a critical point.

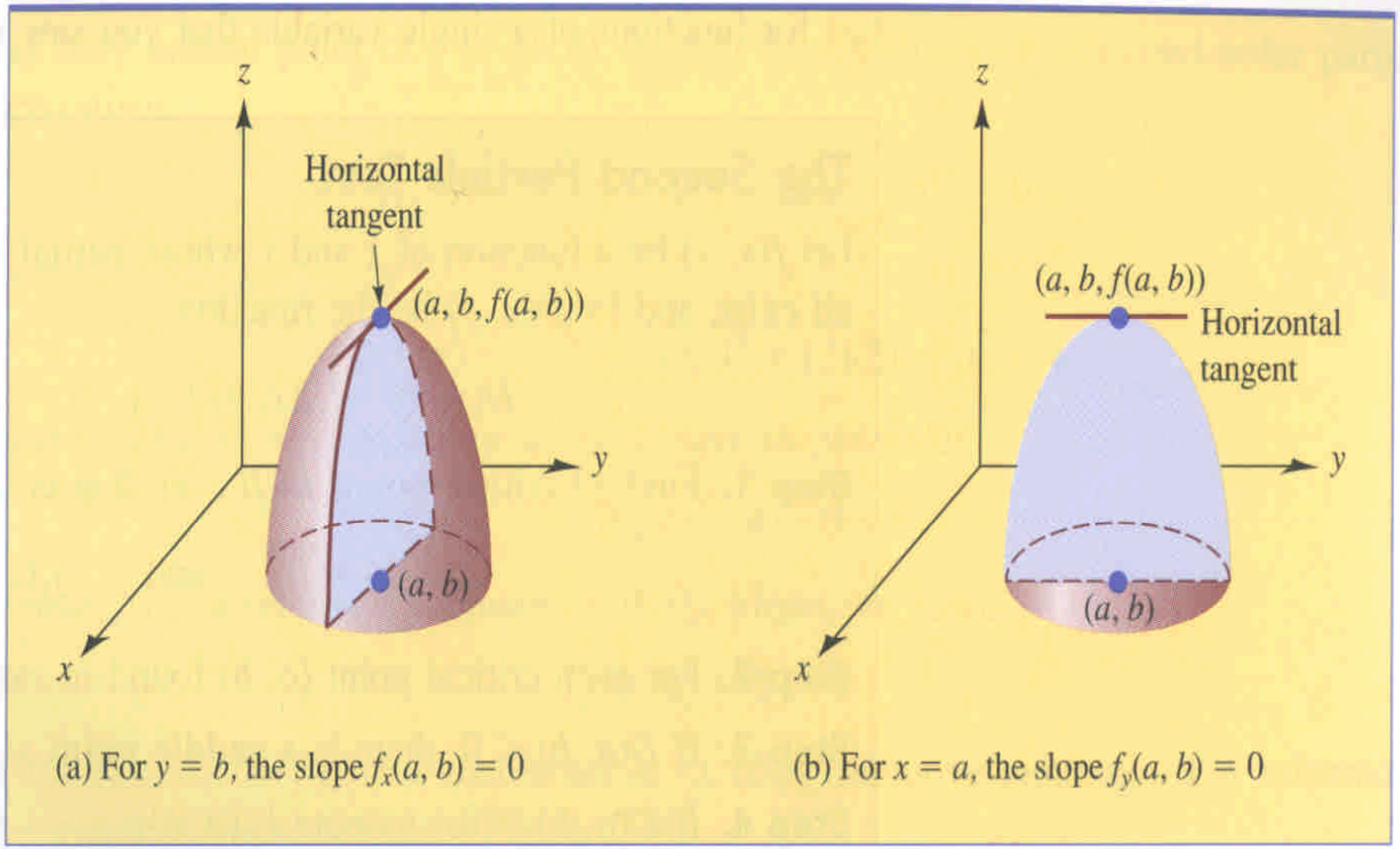

FIGURE 7.12 The partial derivatives are zero at a relative extremum.

Here is a more precise statement of the situation.

Critical points and relative extrema (p590)

Notes

- Although all the relative extrema of a function must occur at critical points, not every critical point of a function corresponds a relative extremum.
- Example: saddle point (p590), Figure 7.13.

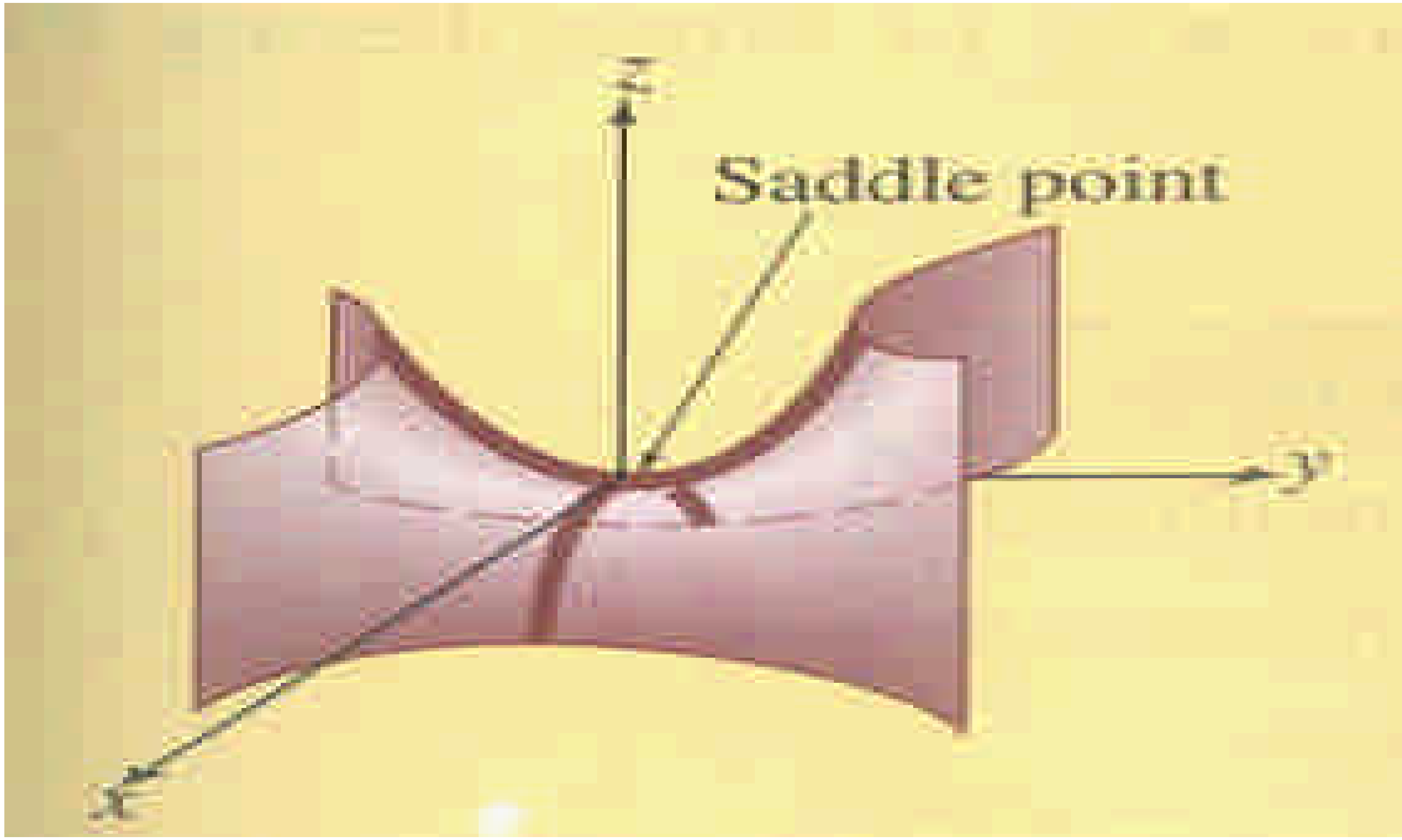

FIGURE 7.13 The saddle surface $z = y^2 - x^2$.

The second partial test (p590)

Notes

- There is a saddle point at the critical point (a,b) only when the quantity D in the second partial test is negative.
- The tabular summary of the conclusions of the second partial test (p591).

Example 7.3.1

Find all critical points for the function $f(x, y) = x^2 + y^2$ and classify each as a relative maximum, a relative minimum, or a saddle point.

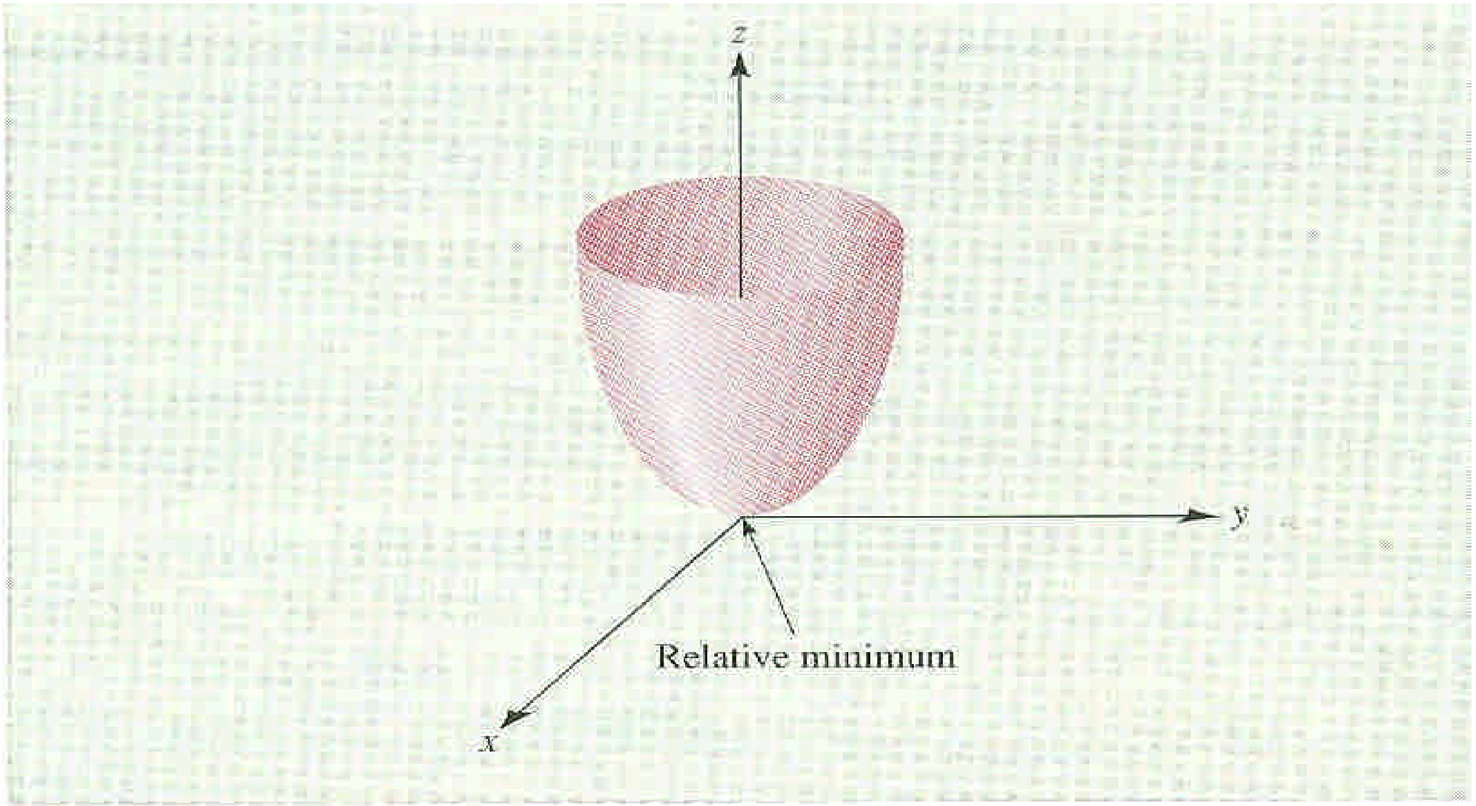

FIGURE 7.14 The surface $z = x^2 + y^2$ with a relative minimum at $(0, 0)$.

Example 7.3.2

Find all critical points for the function $f(x,y) = 12x - x^3 - xy^2$ and classify each as a relative maximum, a relative minimum, or a saddle point.

Example 7.3.3

Find all critical points for the function $f(x,y)=x^3-y^3+6xy$ and classify each as a relative maximum, a relative minimum, or a saddle point.

Example 7.3.4

- The only grocery store in a small rural community carries two brands of frozen apple juice,
    - a local brand that it obtains at the cost of 30 cents per can and
    - a well-known national brand that it obtains at the cost of 40 cents per can.

Example 7.3.4

- The grocer estimates that if the local brand is sold for x cents per can and the national brand for y cents per can, then
  - approximately 70-5x+4y cans of the local brand and
  - 80+6x-7y cans of the national brand will be sold every day.
- **Q**: How should the grocer price each brand to maximize the total daily profit from the sale of cat food?

Example 7.3.5

- The business manager for Acme Corporation plots a grid on a map of the region Acme serves and determines that the company's three most important customers are located serves and determines that the company's three most important customers are located at points $A(1,5)$, $B(0,0)$, and $C(8,0)$, where units are in miles.
- At what point $W(x,y)$ should a warehouse be located in order to minimize the sum of the squares of the distances from W to A, B, and C (see Figure 7.15)?

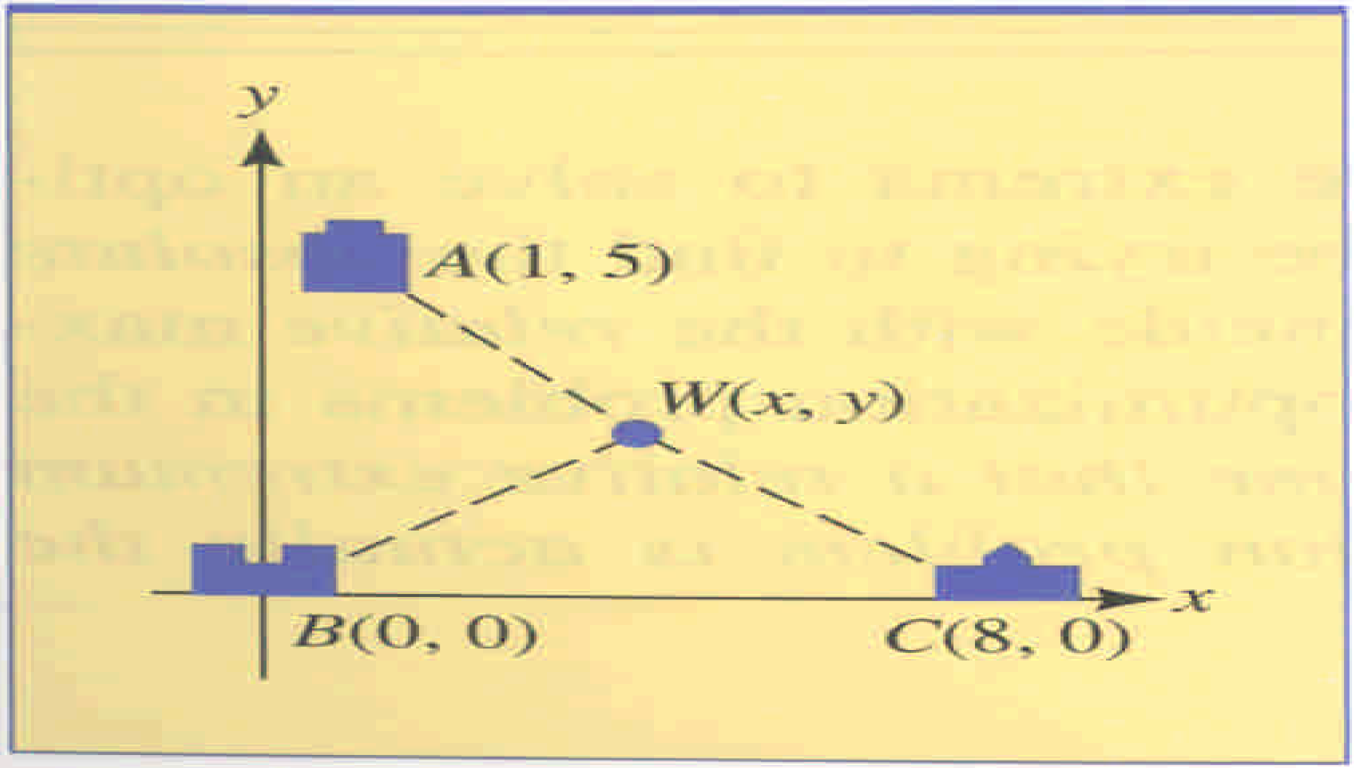

FIGURE 7.15 Locations of businesses A, B, and C and warehouse W.

Section 7.4 The method of Least-Squares (最小平方法)

Motivation for Least-Squares (LS) method

- You have seen applied functions, many of which are derived from published research, and you may have wondered
  - **how researchers come up with such functions**.
- A common procedure for associating a function with an observed physical phenomenon is to
  - Gather data,
  - Plot the data on the graph, and
  - Find a function whose graph "best fits" the data in some mathematically meaningful way.
- LS method (or regression analysis) is such a procedure to meet the above process.

LS methods

- The first step of LS method is to draw the scatter diagram, see Figure 7.16 (p601), in order to decide what type of function to try.
- Once the type of function has been chosen, the next step is to determine the particular function of this type whose graph is "closest" to the given set of points.
- Least-squares criterion, see Figure 7.17 (p601).

Example 7.4.1

- Use the LS criterion to find the equation of the line that is closest to the three points (1,1), (2,3), and (4,3).

Ans: $y=4x/7+1$

Note: Formula of the LS line is seen on p603.

Example 7.4.3

- A college admissions officer has compiled these data relating students' high school and college grade-point averages (GPA), see table on p604.
- Find the equation of the LS line for these data and use it to predict the college GPA of a student whose high school GPA is 3.7?

Example 7.4.4

- A manufacturer gathers these data relating the level of producing x (hundred units) of a particular commodity to the demand price p (dollars per unit) at which all x units will be sold, see table on p605.
- Q1: Plot a scatter diagram for the data on a graph with production level on the x axis and demand price on the y axis.

Example 7.4.4

- Q2: Suppose the exponential function is recommended for the demand function, based on Q1, modify the LS procedure to find a cure of the form $p = Ae^{mx}$ that best fits the data in the table.
- Q3: Use the exponential demand function you found in Q2 to predict the revenue the manufacturer should expect if 4000 (x=40) units are produced.

Skip Section 7.5

We may come back to this section later, if necessary

Section 7.6 Double integrals (雙重積分)

Preliminaries

- Recall the integral of f(x)
- For f(x,y), we shall integrate f(x,y) by holding one variable fixed and integrating with respect to the other, see examples on p628.
- In general, partially integrating a function f(x,y) w.r.t. x results in a function of y alone, which can then be integrated as a function of a single variable, thus producing what we call an iterated integral.

The double integral over a rectangular region (p629)

Examples 7.6.1-2

Double integrals over rectangular regions

Double integrals over nonrectangular regions (P631)

Examples 7.6.3-5 (p632-

Double integrals over nonrectangular regions (P631)

- Double integrals can also be defined over nonrectangular regions.
- Efficient procedure for describing certain such regions in terms of inequalities.
  - Vertical cross sections, see Example 7.6.3 and Figure 7.27 (p632).
  - Horizontal cross section, see Example 7.6.4 and Figure 7.28 (p633).

Limits of integration for double integrals (p634)

Example 7.6.5 (p634)

Area formula (p636)

Example 7.6.6 (p636)

Volume as a double integral (p637)

Example 7.6.7 (p637)

Average value of a function f(x,y)

Example 7.6.8 (p639)