Calculus of Several Variables

8

- Functions of Several Variables
- Partial Derivatives
- Maxima and Minima of Function of Several Variables
- The Method of Least Squares
- Constrained Maxima and Minima and

  the Method of Lagrange Multipliers
- Double Integrals

8.1 Functions of Two Variables

A real-valued function of two variables, f, consists of

1. A set A of ordered pairs of real numbers (x, y) called the domain of the function.

2. A rule that associates with each ordered pair in the domain of f one and only one real number, denoted by $z = f(x, y)$.

EXAMPLE 1 Let f be the function defined by

$$f(x, y) = x + xy + y^2 + 2$$

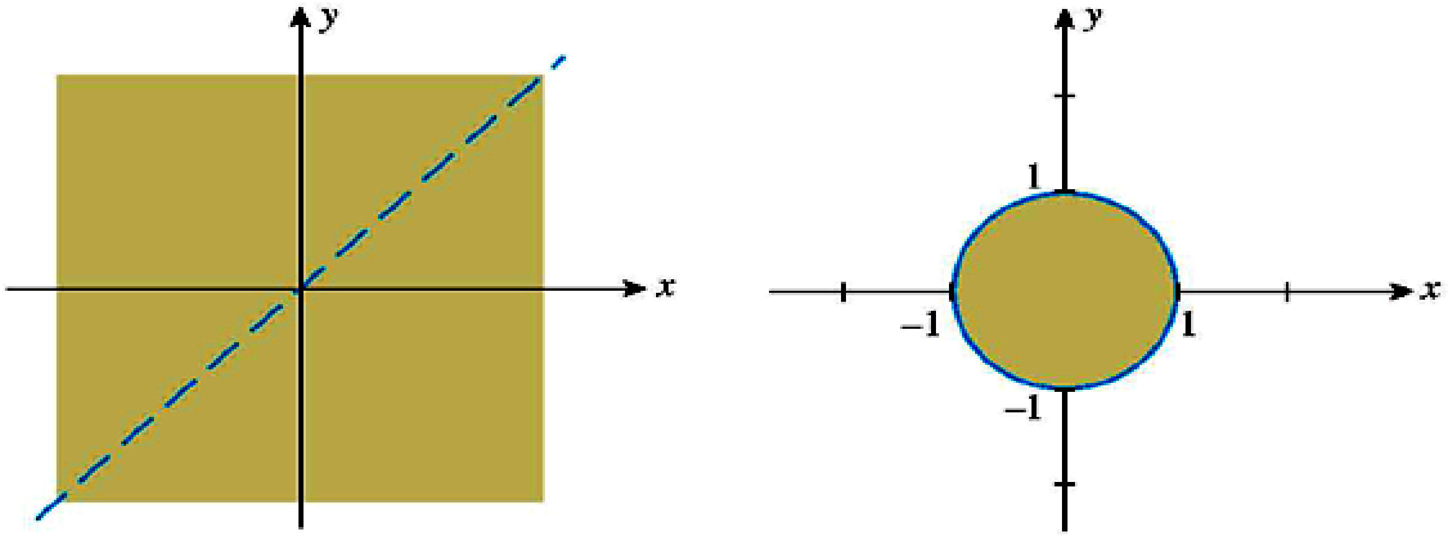
Compute $f(0, 0)$, $f(1, 2)$, and $f(2, 1)$.

Solution We have

$$f(0, 0) = 0 + (0)(0) + 0^2 + 2 = 2$$

$$f(1, 2) = 1 + (1)(2) + 2^2 + 2 = 9$$

$$f(2, 1) = 2 + (2)(1) + 1^2 + 2 = 7$$



(a) Domain of g (b) Domain of h

FIGURE 1

EXAMPLE 2 Find the domain of each of the following functions.

a. $f(x, y) = x^2 + y^2$ **b.** $g(x, y) = \dfrac{2}{x - y}$ **c.** $h(x, y) = \sqrt{1 - x^2 - y^2}$

Solution

a. $f(x, y)$ is defined for all real values of x and y, so the domain of the function f is the set of all points (x, y) in the xy-plane.

b. $g(x, y)$ is defined for all $x \neq y$, so the domain of the function g is the set of all points in the xy-plane except those lying on the line $y = x$ (Figure 1a).

c. We require that $1 - x^2 - y^2 \geq 0$ or $x^2 + y^2 \leq 1$, which is just the set of all points (x, y) lying on and inside the circle of radius 1 with center at the origin (Figure 1b). ■

Ex. Let f be the function defined by

$$f(x,y)=3x^2y-2+y^3.$$

Find $f(0,3)$ and $f(2,-1)$.

$$f(0,3)=3(0)^2(3)-2+(3)^3$$

$$=25$$

$$f(2,-1)=3(2)^2(-1)-2+(-1)^3$$

$$=-15$$

Ex. Find the domain of each function

a. $f(x, y) = 3x - 2y^2$

Since $f(x, y)$ is defined for all real values of x and y, the domain of f is the set of all points (x, y) in the xy – plane.

b. $g(x, y) = \dfrac{x}{2x + y - 3}$

$g(x, y)$ is defined as long as $2x + y - 3$ is not 0. So the domain is the set of all points (x, y) in the xy – plane except those on the line $y = -2x + 3$.

APPLIED EXAMPLE 3 Revenue Functions Acrosonic manufactures a bookshelf loudspeaker system that may be bought fully assembled or in a kit. The demand equations that relate the unit prices, p and q, to the quantities demanded weekly, x and y, of the assembled and kit versions of the loudspeaker systems are given by

$$p = 300 - \frac{1}{4}x - \frac{1}{8}y \quad \text{and} \quad q = 240 - \frac{1}{8}x - \frac{3}{8}y$$

a. What is the weekly total revenue function $R(x, y)$?

b. What is the domain of the function R?

Solution

a. The weekly revenue realizable from the sale of x units of the assembled speaker systems at p dollars per unit is given by xp dollars. Similarly, the weekly revenue realizable from the sale of y units of the kits at q dollars per unit is given by yq dollars. Therefore, the weekly total revenue function R is given by

$$\begin{aligned} R(x, y) &= xp + yq \\ &= x\left(300 - \frac{1}{4}x - \frac{1}{8}y\right) + y\left(240 - \frac{1}{8}x - \frac{3}{8}y\right) \\ &= -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 300x + 240y \end{aligned}$$

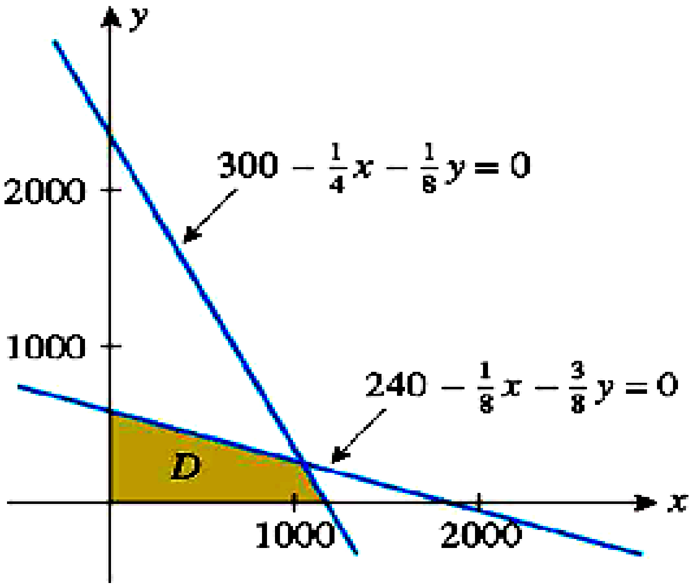


FIGURE 2
The domain of $R(x, y)$

b. To find the domain of the function R, let's observe that the quantities x, y, p, and q must be nonnegative. This observation leads to the following system of linear inequalities:

$$\begin{aligned} 300 - \frac{1}{4}x - \frac{1}{8}y &\geq 0 \\ 240 - \frac{1}{8}x - \frac{3}{8}y &\geq 0 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

The domain of the function R is sketched in Figure 2.

APPLIED EXAMPLE 4 Home Mortgage Payments The monthly payment that amortizes a loan of A dollars in t years when the interest rate is r per year is given by

$$P = f(A, r, t) = \frac{Ar}{12\left[1 - \left(1 + \frac{r}{12}\right)^{-12t}\right]}$$

Find the monthly payment for a home mortgage of $270,000 to be amortized over 30 years when the interest rate is 10% per year.

Solution Letting $A = 270{,}000$, $r = 0.1$, and $t = 30$, we find the required monthly payment to be

$$P = f(270{,}000, 0.1, 30) = \frac{270{,}000(0.1)}{12\left[1 - \left(1 + \frac{0.1}{12}\right)^{-360}\right]} \approx 2369.44$$

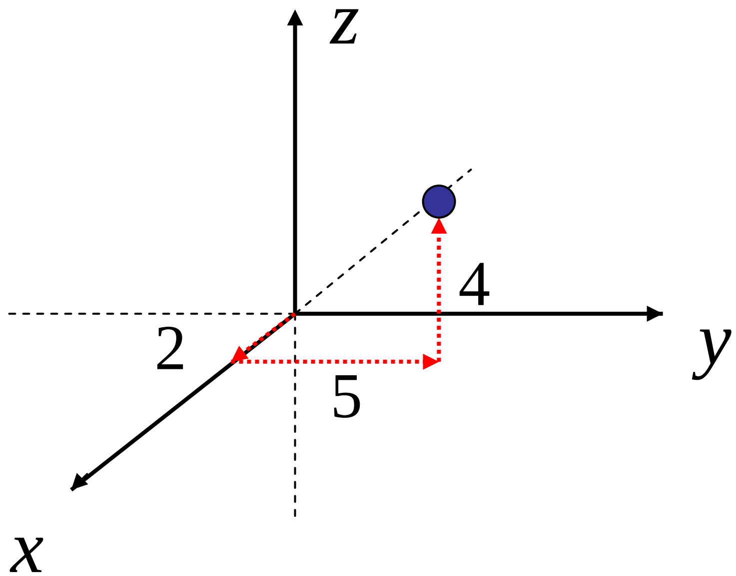
or approximately $2369.44.

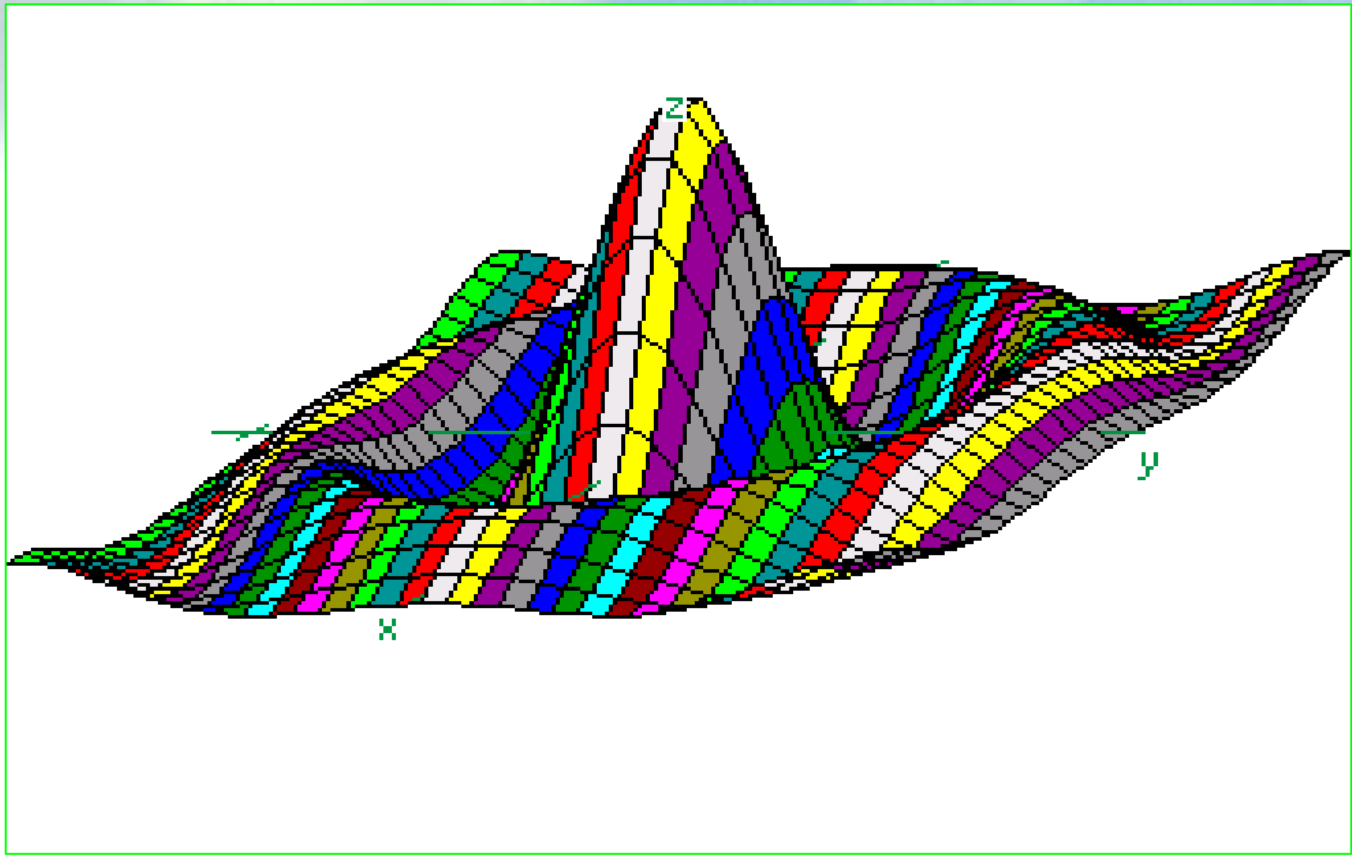
Amortise:分期償還(債務等); Mortgage:抵押

Graphs of Functions of Two Variables

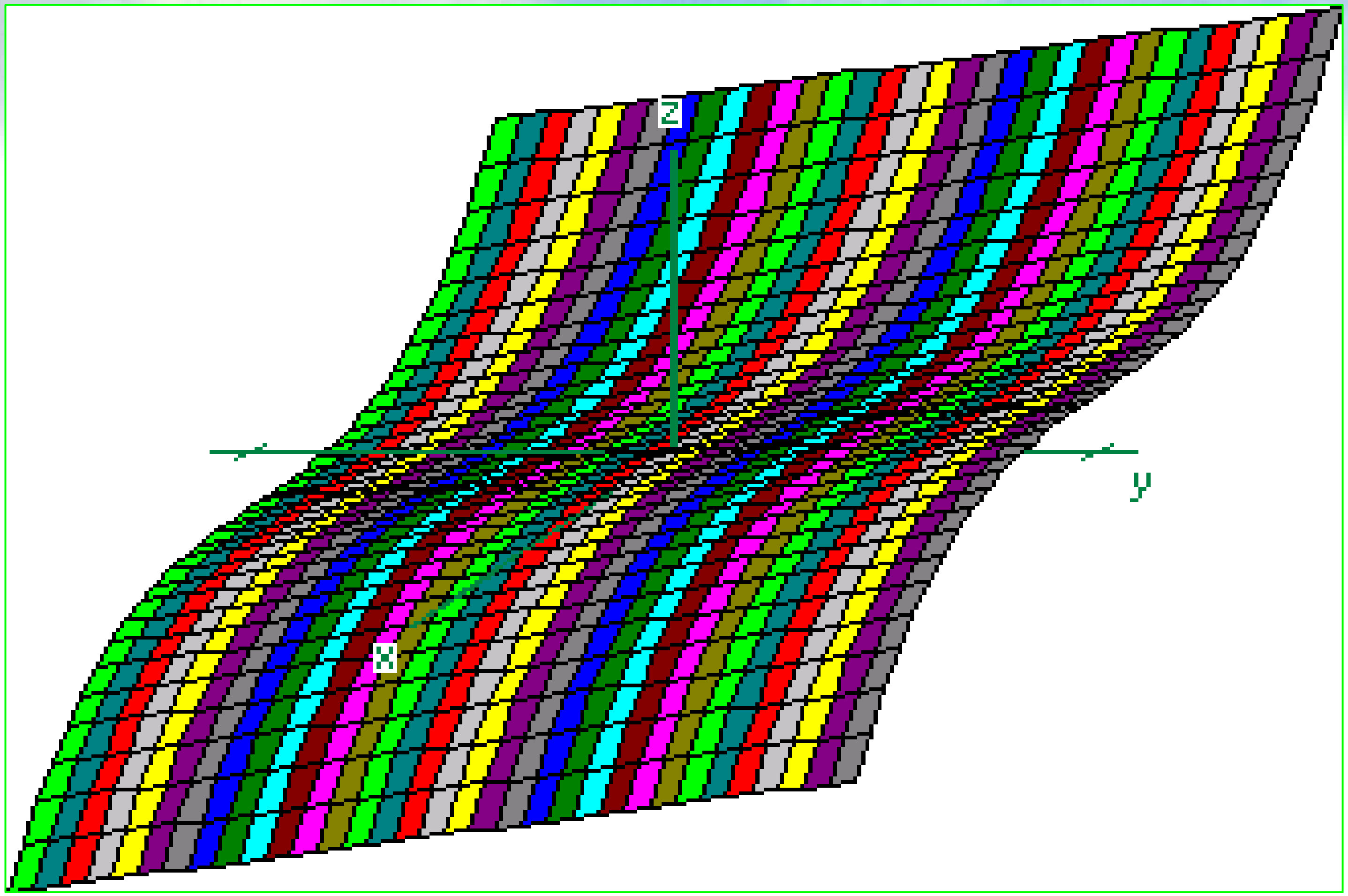
Three-dimensional coordinate system: (x, y, z)

Ex. Plot (2, 5, 4)





The graph of $z = \frac{\sin\left(\sqrt{x^2 + y^2}\right)}{\sqrt{x^2 + y^2}}$



The graph of $z = y - x^3$

Level Curves

- $f(x, y)$ is a function of two variables.

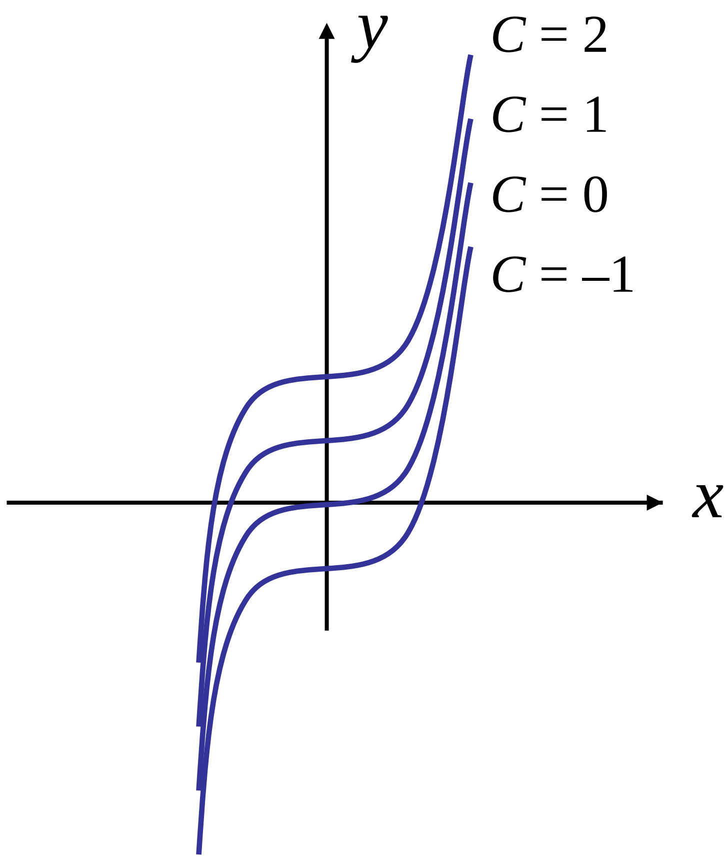
- If c is some value of the function f, a trace of the graph of $z = f(x, y) = c$ projected in the xy-plane is called a *level curve*.

- A *contour map (輪廓圖)* is created by drawing several values of *c*.

Ex. Sketch the level curves for the function $f(x, y) = y - x^3$ corresponding to $z = -1, 0, 1, 2$.

We have the family of curves

$$y = x^3 + C$$



EXAMPLE 5 Sketch a contour map for the function $f(x, y) = x^2 + y^2$.

Solution The level curves are the graphs of the equation $x^2 + y^2 = c$ for nonnegative numbers c. Taking $c = 0, 1, 4, 9$, and 16, for example, we obtain

$$c = 0: x^2 + y^2 = 0$$

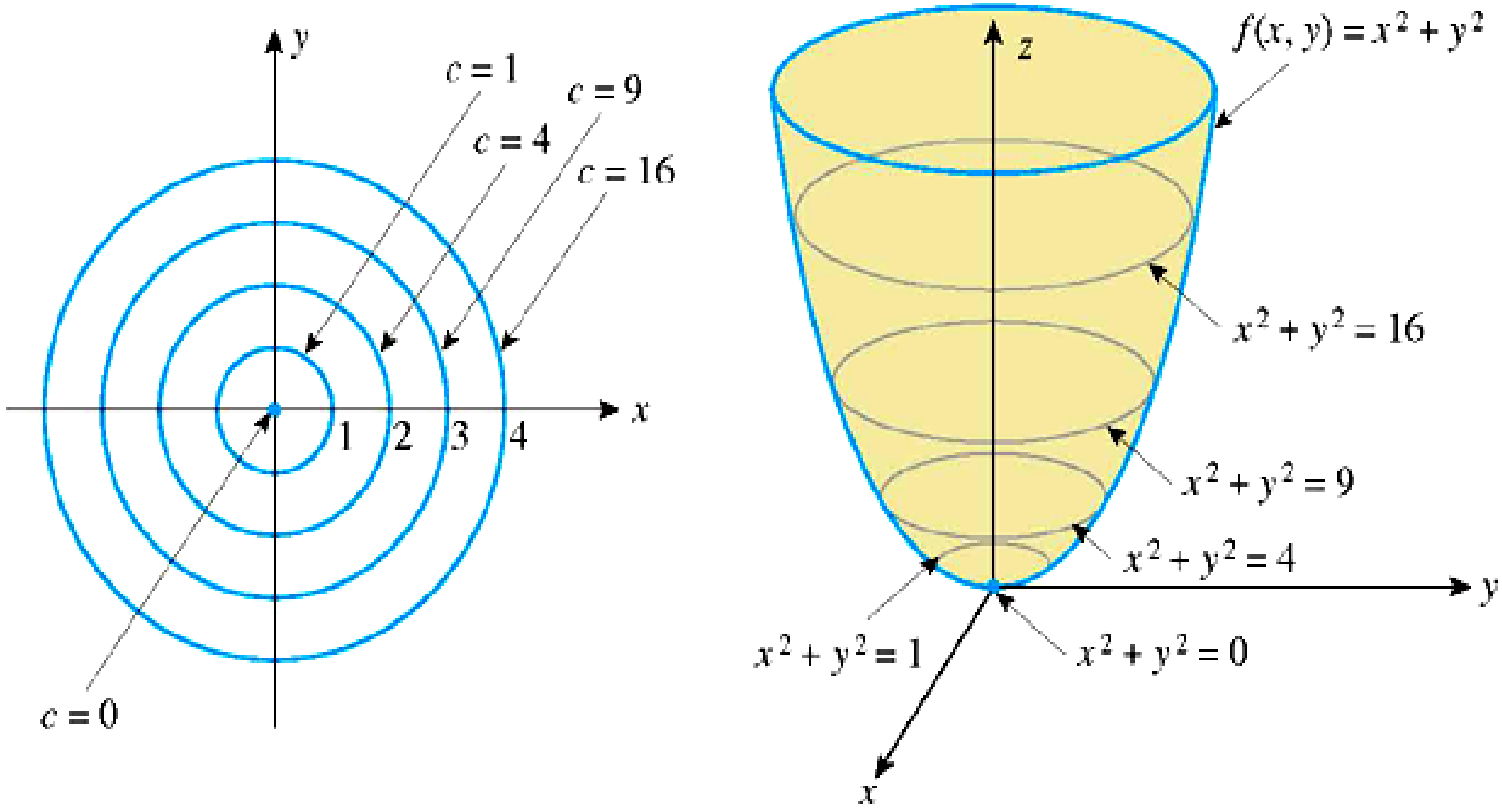
$$c = 1: x^2 + y^2 = 1$$

$$c = 4: x^2 + y^2 = 4 = 2^2$$

$$c = 9: x^2 + y^2 = 9 = 3^2$$

$$c = 16: x^2 + y^2 = 16 = 4^2$$

The five level curves are concentric circles with center at the origin and radius given by $r = 0, 1, 2, 3$, and 4, respectively (Figure 9a). A sketch of the graph of $f(x, y) = x^2 + y^2$ is included for your reference in Figure 9b. ■



(a) Level curves of $f(x, y) = x^2 + y^2$

(b) The graph of $f(x, y) = x^2 + y^2$

FIGURE 9

EXAMPLE 6 Sketch the level curves for the function $f(x, y) = 2x^2 - y$ corresponding to $z = -2, -1, 0, 1$, and 2.

Solution The level curves are the graphs of the equation $2x^2 - y = k$ or $y = 2x^2 - k$ for $k = -2, -1, 0, 1$, and 2. The required level curves are shown in Figure 10. ■

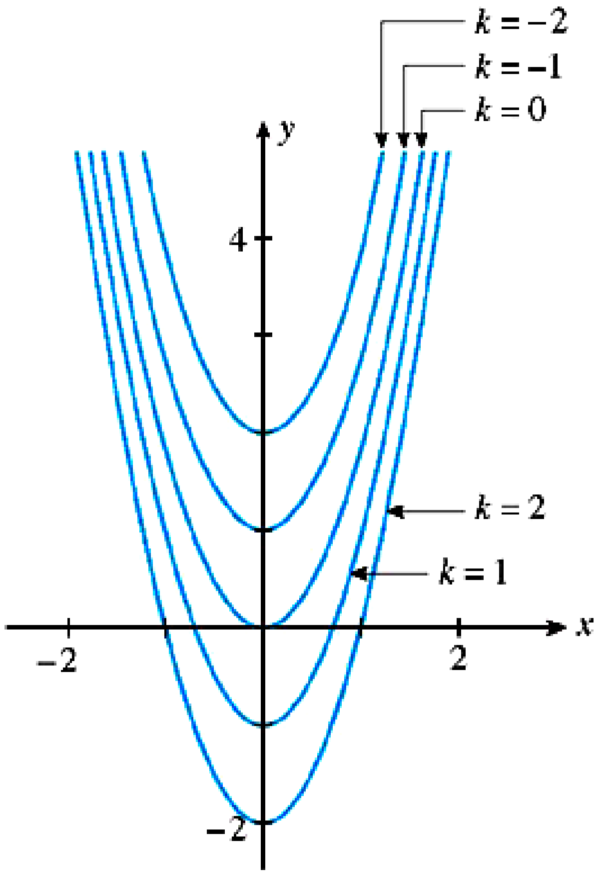


FIGURE 10
Level curves for $f(x, y) = 2x^2 - y$

8.2 First Partial Derivatives of $f(x, y)$.

$f(x, y)$ is a function of two variables. The first partial derivative of f with respect to x at a point (x, y) is

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

provided the limit exists.

The first partial derivative with respect to y at (x, y) is

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

provided the limit exists.

Notes

- Figure 12: approach a point in the plane from infinitely many directions.
- Figure 13: The curve C is formed by the intersection of the plane y=b with the surface z=f(x,y).
- Figure 14: The curve C is formed by the intersection of the plane x=a with the surface z=f(x,y).
- First partial derivative of f with respect to x at (a,b)
  - This derivative, obtaining by keeping the variable y fixed and differentiating function f(x,b) w.r.t. and evaluating at x=a.

Ex. Compute the first partial derivatives of the function: $f(x, y) = 3x^2 y + x \ln y$

$$f_x = 6xy + \ln y$$

$$f_y = 3x^2 + x\left(\frac{1}{y}\right)$$

Ex. Given the function $g(x, y)$, compute g_y.

$$g(x, y) = e^{xy^2 + y}$$

$$g_y = (2xy + 1)e^{xy^2 + y}$$

EXAMPLE 1 Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the function

$$f(x, y) = x^2 - xy^2 + y^3$$

What is the rate of change of the function f in the x-direction at the point (1, 2)? What is the rate of change of the function f in the y-direction at the point (1, 2)?

Solution To compute $\frac{\partial f}{\partial x}$, think of the variable y as a constant and differentiate the resulting function of x with respect to x. Let's write

$$f(x, y) = x^2 - xy^2 + y^3$$

where the variable y to be treated as a constant is shown in color. Then,

$$\frac{\partial f}{\partial x} = 2x - y^2$$

To compute $\frac{\partial f}{\partial y}$, think of the variable x as being fixed—that is, as a constant—and differentiate the resulting function of y with respect to y. In this case,

$$f(x, y) = x^2 - xy^2 + y^3$$

so that

$$\frac{\partial f}{\partial y} = -2xy + 3y^2$$

The rate of change of the function f in the x-direction at the point (1, 2) is given by

$$f_x(1,2) = \left.\frac{\partial f}{\partial x}\right|_{(1,2)} = 2(1) - 2^2 = -2$$

That is, f decreases 2 units for each unit increase in the x-direction, y being kept constant ($y = 2$). The rate of change of the function f in the y-direction at the point (1, 2) is given by

$$f_y(1,2) = \left.\frac{\partial f}{\partial y}\right|_{(1,2)} = -2(1)(2) + 3(2)^2 = 8$$

That is, f increases 8 units for each unit increase in the y-direction, x being kept constant ($x = 1$). ■

EXAMPLE 2 Compute the first partial derivatives of each function.

a. $f(x, y) = \dfrac{xy}{x^2 + y^2}$ **b.** $g(s, t) = (s^2 - st + t^2)^5$

c. $h(u, v) = e^{u^2 - v^2}$ **d.** $f(x, y) = \ln(x^2 + 2y^2)$

Solution

a. To compute $\dfrac{\partial f}{\partial x}$, think of the variable y as a constant. Thus,

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

so that, upon using the quotient rule, we have

$$\frac{\partial f}{\partial x} = \frac{(x^2 + y^2)y - xy(2x)}{(x^2 + y^2)^2} = \frac{x^2y + y^3 - 2x^2y}{(x^2 + y^2)^2}$$

$$= \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}$$

upon simplification and factorization. To compute $\dfrac{\partial f}{\partial y}$, think of the variable x as a constant. Thus,

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

so that, upon using the quotient rule once again, we obtain

$$\frac{\partial f}{\partial y} = \frac{(x^2 + y^2)x - xy(2y)}{(x^2 + y^2)^2} = \frac{x^3 + xy^2 - 2xy^2}{(x^2 + y^2)^2}$$

$$= \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}$$

b. To compute $\partial g/\partial s$, we treat the variable t as if it were a constant. Thus,

$$g(s, t) = (s^2 - st + t^2)^5$$

Using the general power rule, we find

$$\frac{\partial g}{\partial s} = 5(s^2 - st + t^2)^4 \cdot (2s - t)$$

$$= 5(2s - t)(s^2 - st + t^2)^4$$

To compute $\partial g/\partial t$, we treat the variable s as if it were a constant. Thus,

$$g(s, t) = (s^2 - st + t^2)^5$$

$$\frac{\partial g}{\partial t} = 5(s^2 - st + t^2)^4(-s + 2t)$$

$$= 5(2t - s)(s^2 - st + t^2)^4$$

c. To compute $\partial h/\partial u$, think of the variable v as a constant. Thus,

$$h(u, v) = e^{u^2 - v^2}$$

Using the chain rule for exponential functions, we have

$$\frac{\partial h}{\partial u} = e^{u^2 - v^2} \cdot 2u$$

$$= 2ue^{u^2 - v^2}$$

Next, we treat the variable u as if it were a constant,

$$h(u, v) = e^{u^2 - v^2}$$

and we obtain

$$\frac{\partial h}{\partial v} = e^{u^2 - v^2} \cdot (-2v)$$

$$= -2ve^{u^2 - v^2}$$

d. To compute $\partial f/\partial x$, think of the variable y as a constant. Thus,

$$f(x, y) = \ln(x^2 + 2y^2)$$

so that the chain rule for logarithmic functions gives

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + 2y^2}$$

Next, treating the variable x as if it were a constant, we find

$$f(x, y) = \ln(x^2 + 2y^2)$$

$$\frac{\partial f}{\partial y} = \frac{4y}{x^2 + 2y^2}$$

EXAMPLE 3 Compute the first partial derivatives of the function

$$w = f(x, y, z) = xyz - xe^{yz} + x \ln y$$

Solution Here we have a function of three variables, x, y, and z, and we are required to compute

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}$$

To compute f_x, we think of the other two variables, y and z, as fixed, and we differentiate the resulting function of x with respect to x, thereby obtaining

$$f_x = yz - e^{yz} + \ln y$$

To compute f_y, we think of the other two variables, x and z, as constants, and we differentiate the resulting function of y with respect to y. We then obtain

$$f_y = xz - xze^{yz} + \frac{x}{y}$$

Finally, to compute f_z, we treat the variables x and y as constants and differentiate the function f with respect to z, obtaining

$$f_z = xy - xye^{yz}$$

The Cobb-Douglas Production Function

$$f(x, y) = ax^b y^{1-b}$$

- a and b are positive constants with $0 < b < 1$.
- x stands for the money spent on labor.
- y stands for the cost of capital equipment.

f_x is the *marginal productivity of labor*.
f_y is the *marginal productivity of capital*.

Ex. A certain production function is given by

$$f(x,y) = 28x^{1/4}y^{3/4}$$

units, when x units of labor and y units of capital are used. Find the marginal productivity of capital when labor = 81 units and capital = 256 units.

$$f_y = 21x^{1/4}y^{-1/4} = 21\left(\frac{x}{y}\right)^{1/4}$$

$$= 21\left(\frac{81}{256}\right)^{1/4} = 21\left(\frac{3}{4}\right) = 15.75$$

So 15.75 units per unit increase in capital expenditure.

APPLIED EXAMPLE 4 Marginal Productivity A certain country's production in the early years following World War II is described by the function

$$f(x, y) = 30x^{2/3}y^{1/3}$$

units, when x units of labor and y units of capital were used.

a. Compute f_x and f_y.

b. What is the marginal productivity of labor and the marginal productivity of capital when the amounts expended on labor and capital are 125 units and 27 units, respectively?

c. Should the government have encouraged capital investment rather than increasing expenditure on labor to increase the country's productivity?

Solution

a. $f_x = 30 \cdot \frac{2}{3}x^{-1/3}y^{1/3} = 20\left(\frac{y}{x}\right)^{1/3}$

$f_y = 30x^{2/3} \cdot \frac{1}{3}y^{-2/3} = 10\left(\frac{x}{y}\right)^{2/3}$

b. The required marginal productivity of labor is given by

$$f_x(125, 27) = 20\left(\frac{27}{125}\right)^{1/3} = 20\left(\frac{3}{5}\right)$$

or 12 units per unit increase in labor expenditure (capital expenditure is held constant at 27 units). The required marginal productivity of capital is given by

$$f_y(125, 27) = 10\left(\frac{125}{27}\right)^{2/3} = 10\left(\frac{25}{9}\right)$$

or $27\frac{7}{9}$ units per unit increase in capital expenditure (labor outlay is held constant at 125 units).

c. From the results of part (b), we see that a unit increase in capital expenditure resulted in a much faster increase in productivity than a unit increase in labor expenditure would have. Therefore, the government should have encouraged increased spending on capital rather than on labor during the early years of reconstruction. ■

Substitute and complementary commodities

- Two commodities are substitute (competitive) commodities if
  - a decrease in the demand for one results in an increase in the demand for the other, such as coffee vs. tea.
- Two commodities are complementary commodities if
  - a decrease in the demand for one results in a decrease in the demand for the other, such as automobiles vs. tires.

Substitute and Complementary Commodities

Two commodities *A* and *B* are *substitute commodities* if

$$\frac{\partial f}{\partial q} > 0 \quad \text{and} \quad \frac{\partial g}{\partial p} > 0$$

Two commodities *A* and *B* are *complementary commodities* if

$$\frac{\partial f}{\partial q} < 0 \quad \text{and} \quad \frac{\partial g}{\partial p} < 0$$

APPLIED EXAMPLE 5 Substitute and Complementary Commodities Suppose that the daily demand for butter is given by

$$x = f(p, q) = \frac{3q}{1 + p^2}$$

and the daily demand for margarine is given by

$$y = g(p, q) = \frac{2p}{1 + \sqrt{q}} \qquad (p > 0, q > 0)$$

where p and q denote the prices per pound (in dollars) of butter and margarine, respectively, and x and y are measured in millions of pounds. Determine whether these two commodities are substitute, complementary, or neither.

Solution We compute

$$\frac{\partial f}{\partial q} = \frac{3}{1 + p^2} \quad \text{and} \quad \frac{\partial g}{\partial p} = \frac{2}{1 + \sqrt{q}}$$

Since

$$\frac{\partial f}{\partial q} > 0 \quad \text{and} \quad \frac{\partial g}{\partial p} > 0$$

for all values of $p > 0$ and $q > 0$, we conclude that butter and margarine are substitute commodities. ■

Second-Order Partial Derivatives

Ex. Compute the second-order partial derivatives of the function: $f(x,y) = x^2y^3 + x^5 - x\ln y$

First partials

$$f_x = 2xy^3 + 5x^4 - \ln y$$

$$f_y = 3x^2y^2 - \frac{x}{y}$$

Second partials

$$f_{xx} = 2y^3 + 20x^3 \qquad f_{yy} = 6x^2y + \frac{x}{y^2}$$

$$f_{xy} = 6xy^2 - \frac{1}{y} \qquad f_{yx} = 6xy^2 - \frac{1}{y}$$

EXAMPLE 6 Find the second-order partial derivatives of the function

$$f(x, y) = x^3 - 3x^2y + 3xy^2 + y^2$$

Solution The first partial derivatives of f are

$$f_x = \frac{\partial}{\partial x}(x^3 - 3x^2y + 3xy^2 + y^2)$$
$$= 3x^2 - 6xy + 3y^2$$

$$f_y = \frac{\partial}{\partial y}(x^3 - 3x^2y + 3xy^2 + y^2)$$
$$= -3x^2 + 6xy + 2y$$

Therefore,

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(3x^2 - 6xy + 3y^2)$$
$$= 6x - 6y = 6(x - y)$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(3x^2 - 6xy + 3y^2)$$
$$= -6x + 6y = 6(y - x)$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(-3x^2 + 6xy + 2y)$$
$$= -6x + 6y = 6(y - x)$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(-3x^2 + 6xy + 2y)$$
$$= 6x + 2$$

EXAMPLE 7 Find the second-order partial derivatives of the function

$$f(x, y) = e^{xy^2}$$

Solution We have

$$f_x = \frac{\partial}{\partial x}\left(e^{xy^2}\right)$$

$$= y^2 e^{xy^2}$$

$$f_y = \frac{\partial}{\partial y}\left(e^{xy^2}\right)$$

$$= 2xye^{xy^2}$$

so the required second-order partial derivatives of f are

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(y^2 e^{xy^2})$$

$$= y^4 e^{xy^2}$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(y^2 e^{xy^2})$$

$$= 2ye^{xy^2} + 2xy^3 e^{xy^2}$$

$$= 2ye^{xy^2}(1 + xy^2)$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(2xye^{xy^2})$$

$$= 2ye^{xy^2} + 2xy^3 e^{xy^2}$$

$$= 2ye^{xy^2}(1 + xy^2)$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(2xye^{xy^2})$$

$$= 2xe^{xy^2} + (2xy)(2xy)e^{xy^2}$$

$$= 2xe^{xy^2}(1 + 2xy^2)$$

8.3 Maxima and minima of functions of several variables

Let f be a function defined on a region R containing (a, b).

$f(a, b)$ is a ***relative maximum*** of f if $f(x, y) \le f(a, b)$ for all (x, y) sufficiently close to (a, b).

$f(a, b)$ is a ***relative minimum*** of f if $f(x, y) \ge f(a, b)$ for all (x, y) sufficiently close to (a, b).

*If the inequalities hold for all (x, y) in the domain of f, then the points are ***absolute extrema***.

Notes

- Figure 16 (p559)
- Figure 17 (560)
- Figure 18 (p561): Saddle point (馬鞍點)
- Figure 19 (p561)
- What are conditions required for the relative extrema?

Critical point of f

- A critical point of f is a point (a,b) in the domain of f such that both $f_x = 0, \quad \text{and} \quad f_y = 0$ at the point (a,b) or at least one of the partial derivatives does not exist. .

Determining Relative Extrema

1. Find all the critical points by solving the system

$$f_x = 0, \quad \text{and} \quad f_y = 0$$

2. The 2nd Derivative Test: Compute

$$D(x, y) = f_{xx} f_{yy} - f_{xy}^2$$

$D(a,b)$	$f_{xx}(a,b)$	Interpretation
+	+	Relative min. at (a, b)
+	–	Relative max. at (a, b)
–		Neither max. nor min. at (a, b)
0		Test is inconclusive

Ex. Determine the relative extrema of the function

$$f(x,y) = 2x - x^2 - y^2$$

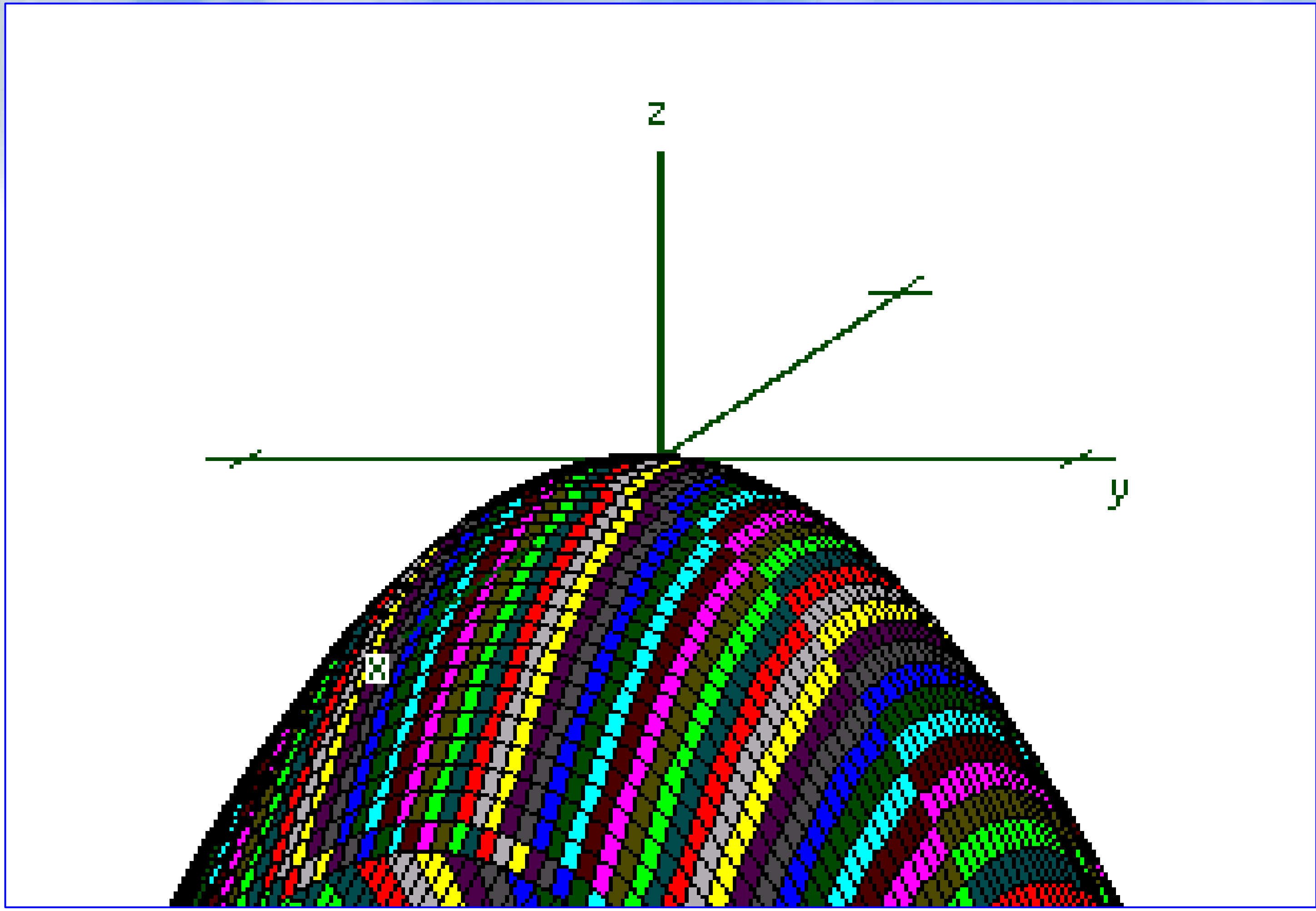
So the only critical point is (1, 0).

$$f_x = 2 - 2x = 0 \qquad f_y = -2y = 0$$

$$f_{xx} = f_{yy} = -2, \qquad f_{xy} = 0$$

$$D(1,0) = (-2)(-2) - 0^2 = 4 > 0 \quad \text{and} \quad f_{xx}(1,0) = -2 < 0$$

So $f(1,0) = 1$ is a relative maximum



The graph of $z = 2x - x^2 - y^2$

EXAMPLE 1 Find the relative extrema of the function

$$f(x, y) = x^2 + y^2$$

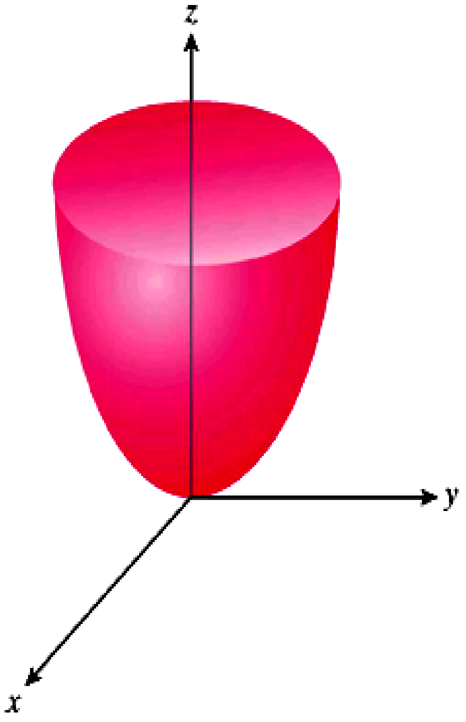


FIGURE 20
The graph of $f(x, y) = x^2 + y^2$

Solution We have

$$f_x = 2x$$
$$f_y = 2y$$

To find the critical point(s) of f, we set $f_x = 0$ and $f_y = 0$ and solve the resulting system of simultaneous equations

$$2x = 0$$
$$2y = 0$$

obtaining $x = 0$, $y = 0$, or $(0, 0)$, as the sole critical point of f. Next, we apply the second derivative test to determine the nature of the critical point $(0, 0)$. We compute

$$f_{xx} = 2 \qquad f_{xy} = 0 \qquad f_{yy} = 2$$

and

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - 0 = 4$$

In particular, $D(0, 0) = 4$. Since $D(0, 0) > 0$ and $f_{xx}(0, 0) = 2 > 0$, we conclude that $f(x, y)$ has a relative minimum at the point $(0, 0)$. The relative minimum value, 0, also happens to be the absolute minimum of f. The graph of the function f, shown in Figure 20, confirms these results. ■

EXAMPLE 2 Find the relative extrema of the function

$$f(x, y) = 3x^2 - 4xy + 4y^2 - 4x + 8y + 4$$

Solution We have

$$f_x = 6x - 4y - 4$$
$$f_y = -4x + 8y + 8$$

To find the critical points of f, we set $f_x = 0$ and $f_y = 0$ and solve the resulting system of simultaneous equations

$$\begin{aligned} 6x - 4y &= 4 \\ -4x + 8y &= -8 \end{aligned}$$

Multiplying the first equation by 2 and the second equation by 3, we obtain the equivalent system

$$\begin{aligned} 12x - 8y &= 8 \\ -12x + 24y &= -24 \end{aligned}$$

Adding the two equations gives $16y = -16$, or $y = -1$. We substitute this value for y into either equation in the system to get $x = 0$. Thus, the only critical point of f is the point $(0, -1)$. Next, we apply the second derivative test to determine whether the point $(0, -1)$ gives rise to a relative extremum of f. We compute

$$f_{xx} = 6 \qquad f_{xy} = -4 \qquad f_{yy} = 8$$

and

$$D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = (6)(8) - (-4)^2 = 32$$

Since $D(0, -1) = 32 > 0$ and $f_{xx}(0, -1) = 6 > 0$, we conclude that $f(x, y)$ has a relative minimum at the point $(0, -1)$. The value of $f(x, y)$ at the point $(0, -1)$ is given by

$$f(0, -1) = 3(0)^2 - 4(0)(-1) + 4(-1)^2 - 4(0) + 8(-1) + 4 = 0 \qquad \blacksquare$$

EXAMPLE 3 Find the relative extrema of the function

$$f(x, y) = 4y^3 + x^2 - 12y^2 - 36y + 2$$

Solution To find the critical points of f, we set $f_x = 0$ and $f_y = 0$ simultaneously, obtaining

$$f_x = 2x = 0$$
$$f_y = 12y^2 - 24y - 36 = 0$$

The first equation implies that $x = 0$. The second equation implies that

$$y^2 - 2y - 3 = 0$$
$$(y + 1)(y - 3) = 0$$

—that is, $y = -1$ or 3. Therefore, there are two critical points of the function f—namely, $(0, -1)$ and $(0, 3)$.

Next, we apply the second derivative test to determine the nature of each of the two critical points. We compute

$$f_{xx} = 2 \qquad f_{xy} = 0 \qquad f_{yy} = 24y - 24 = 24(y - 1)$$

Therefore,

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 48(y - 1)$$

For the point $(0, -1)$,

$$D(0, -1) = 48(-1 - 1) = -96 < 0$$

Since $D(0, -1) < 0$, we conclude that the point $(0, -1)$ gives a saddle point of f. For the point $(0, 3)$,

$$D(0, 3) = 48(3 - 1) = 96 > 0$$

Since $D(0, 3) > 0$ and $f_{xx}(0, 3) > 0$, we conclude that the function f has a relative minimum at the point $(0, 3)$. Furthermore, since

$$\begin{aligned} f(0, 3) &= 4(3)^3 + (0)^2 - 12(3)^2 - 36(3) + 2 \\ &= -106 \end{aligned}$$

we see that the relative minimum value of f is -106. ■

APPLIED EXAMPLE 4 Maximizing Profits The total weekly revenue (in dollars) that Acrosonic realizes in producing and selling its bookshelf loudspeaker systems is given by

$$R(x, y) = -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 300x + 240y$$

where x denotes the number of fully assembled units and y denotes the number of kits produced and sold each week. The total weekly cost attributable to the production of these loudspeakers is

$$C(x, y) = 180x + 140y + 5000$$

dollars, where x and y have the same meaning as before. Determine how many assembled units and how many kits Acrosonic should produce per week to maximize its profit.

Solution The contribution to Acrosonic's weekly profit stemming from the production and sale of the bookshelf loudspeaker systems is given by

$$\begin{aligned} P(x, y) &= R(x, y) - C(x, y) \\ &= \left(-\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 300x + 240y\right) - (180x + 140y + 5000) \\ &= -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 120x + 100y - 5000 \end{aligned}$$

To find the relative maximum of the profit function $P(x, y)$, we first locate the critical point(s) of P. Setting $P_x(x, y)$ and $P_y(x, y)$ equal to zero, we obtain

$$P_x = -\frac{1}{2}x - \frac{1}{4}y + 120 = 0$$
$$P_y = -\frac{3}{4}y - \frac{1}{4}x + 100 = 0$$

Solving the first of these equations for y yields

$$y = -2x + 480$$

which, upon substitution into the second equation, yields

$$-\frac{3}{4}(-2x + 480) - \frac{1}{4}x + 100 = 0$$
$$6x - 1440 - x + 400 = 0$$
$$x = 208$$

We substitute this value of x into the equation $y = -2x + 480$ to get

$$y = 64$$

Therefore, the function P has the sole critical point (208, 64). To show that the point (208, 64) is a solution to our problem, we use the second derivative test. We compute

$$P_{xx} = -\frac{1}{2} \qquad P_{xy} = -\frac{1}{4} \qquad P_{yy} = -\frac{3}{4}$$

So,

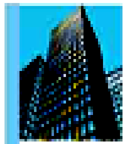
$$D(x, y) = \left(-\frac{1}{2}\right)\left(-\frac{3}{4}\right) - \left(-\frac{1}{4}\right)^2 = \frac{3}{8} - \frac{1}{16} = \frac{5}{16}$$

In particular, $D(208, 64) = 5/16 > 0$.

Since $D(208, 64) > 0$ and $P_{xx}(208, 64) < 0$, the point (208, 64) yields a relative maximum of P. This relative maximum is also the absolute maximum of P. We conclude that Acrosonic can maximize its weekly profit by manufacturing 208 assembled units and 64 kits of their bookshelf loudspeaker systems. The maximum weekly profit realizable from the production and sale of these loudspeaker systems is given by

$$\begin{aligned} P(208, 64) &= -\frac{1}{4}(208)^2 - \frac{3}{8}(64)^2 - \frac{1}{4}(208)(64) \\ &\quad + 120(208) + 100(64) - 5000 \\ &= 10{,}680 \end{aligned}$$

or \$10,680.



APPLIED EXAMPLE 5 **Locating a Television Relay Station Site** A television relay station will serve towns A, B, and C, whose relative locations are shown in Figure 21. Determine a site for the location of the station if the sum of the squares of the distances from each town to the site is minimized.

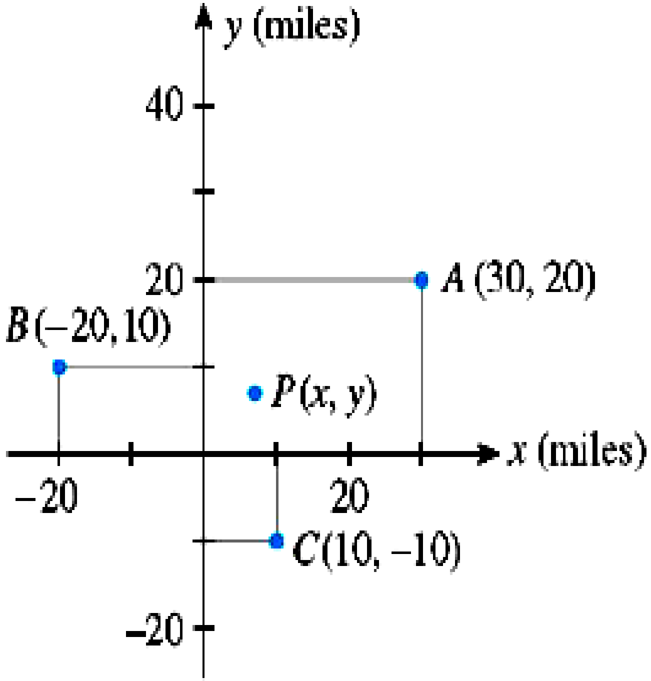


FIGURE 21
Locating a site for a television relay station

Solution Suppose the required site is located at the point $P(x, y)$. With the aid of the distance formula, we find that the square of the distance from town A to the site is

$$(x - 30)^2 + (y - 20)^2$$

The respective distances from towns B and C to the site are found in a similar manner, so the sum of the squares of the distances from each town to the site is given by

$$f(x, y) = (x - 30)^2 + (y - 20)^2 + (x + 20)^2 + (y - 10)^2 + (x - 10)^2 + (y + 10)^2$$

To find the relative minimum of $f(x, y)$, we first find the critical point(s) of f. Using the chain rule to find $f_x(x, y)$ and $f_y(x, y)$ and setting each equal to zero, we obtain

$$f_x = 2(x - 30) + 2(x + 20) + 2(x - 10) = 6x - 40 = 0$$

$$f_y = 2(y - 20) + 2(y - 10) + 2(y + 10) = 6y - 40 = 0$$

from which we deduce that $(\frac{20}{3}, \frac{20}{3})$ is the sole critical point of f. Since

$$f_{xx} = 6 \qquad f_{xy} = 0 \qquad f_{yy} = 6$$

we have

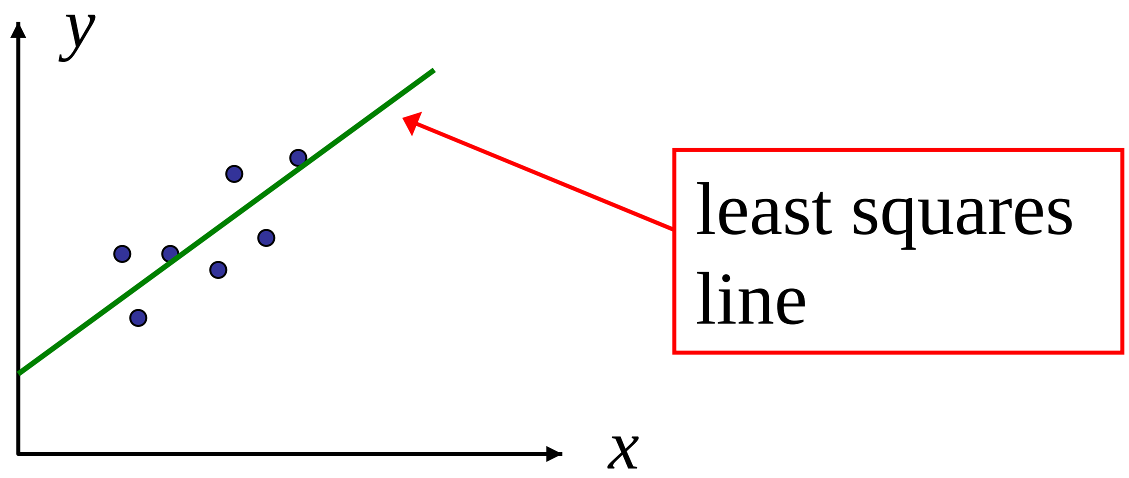
$$D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = (6)(6) - 0 = 36$$

Since $D(\frac{20}{3}, \frac{20}{3}) > 0$ and $f_{xx}(\frac{20}{3}, \frac{20}{3}) > 0$, we conclude that the point $(\frac{20}{3}, \frac{20}{3})$ yields a relative minimum of f. Thus, the required site has coordinates $x = \frac{20}{3}$ and $y = \frac{20}{3}$. ■

8.4 The Method of Least Squares

- The method of *least squares (LS)* is used to determine a straight line that *best* fits a set of data points when the points are scattered about a straight line.

- Functional relationship vs. Statistical relationship



Notes

- Statistical relationships are more common than the functional relationships in practice.
- LS method is quite useful for finding the approximate relationship.
- Figures 22-23 (p570).
- Least-squares line or regression line, obtained from the LS method.

The Method of Least Squares

Given the following n data points:

$$P_1(x_1, y_1),\ P_2(x_2, y_2), \ldots, P_n(x_n, y_n)$$

The *least-squares (regression) line* for the data is given by $y = mx + b$, where m and b satisfy

$$\left(x_1^2 + x_2^2 + \ldots + x_n^2\right)m + \left(x_1 + x_2 + \ldots + x_n\right)b = x_1y_1 + x_2y_2 + \ldots + x_ny_n$$

and $$\left(x_1 + x_2 + \ldots + x_n\right)m + nb = y_1 + y_2 + \ldots + y_n$$

simultaneously. The above two equations are called the normal equations.

Ex. Find the equation of the least-squares line for the data

$$P_1(1,2) \quad P_2(2,3) \quad P_3(3,7)$$

$$\begin{cases} \left(x_1^2 + x_2^2 + x_3^2\right)m + \left(x_1 + x_2 + x_3\right)b = x_1y_1 + x_2y_2 + x_3y_3 \\ \left(x_1 + x_2 + x_3\right)m + nb = y_1 + y_2 + y_3 \end{cases}$$

$$\begin{cases} (1+4+9)m + (1+2+3)b = 2 + 6 + 21 \\ (1+2+3)m + 3b = 2 + 3 + 7 \end{cases}$$

$$\begin{cases} 14m + 6b = 29 \\ 6m + 3b = 12 \end{cases} \qquad m = 2.5,\ b = -1$$

$$\boxed{y = 2.5x - 1}$$

Ex. The following date give the percent of people over age 65 who have high school diplomas.

Year x	0	6	11	16	22	26
Percent with diplomas y	19	25	30	35	44	48

Here, $x = 0$ corresponds to the beginning of the year 1959.

1. Find an equation of the least-square line for the given data.

2. Assuming that this trend continues, what percent of people over age 65 will have high school diplomas at the beginning of the year 2005.

. . .

Solution

1. We need to find the equation of least-squares for the given data.

$$\begin{cases} nb + \left(x_1 + \dots + x_6\right)m = y_1 + \dots + y_6 \\ \left(x_1 + \dots + x_6\right)b + \left(x_1^2 + \dots + x_6^2\right)m = x_1y_1 + \dots + x_6y_6 \end{cases}$$

$$\begin{cases} 1573m + 81b = 3256 \\ 81m + 6b = 201 \end{cases} \Rightarrow m \approx 1.13 \text{ and } b \approx 18.23$$

Therefore, the required least-square line has the equation:

$$\boxed{y = 1.13x + 18.23}$$

. . . .

Solution (cont.)

2. The percent of people over the age of 65 who will have high school diplomas at the beginning of the year 2005 is given by

$$y = f(46) = 1.13\left(46\right) + 18.23 = 70.21$$

or approximately 70.21%.

EXAMPLE 1 Find an equation of the least-squares line for the data

$$P_1(1, 1), \quad P_2(2, 3), \quad P_3(3, 4), \quad P_4(4, 3), \quad P_5(5, 6)$$

Solution Here, we have $n = 5$ and

$$\begin{array}{lllll} x_1 = 1 & x_2 = 2 & x_3 = 3 & x_4 = 4 & x_5 = 5 \\ y_1 = 1 & y_2 = 3 & y_3 = 4 & y_4 = 3 & y_5 = 6 \end{array}$$

so Equation (4) becomes

$$(1 + 4 + 9 + 16 + 25)m + (1 + 2 + 3 + 4 + 5)b = 1 + 6 + 12 + 12 + 30$$

or

$$55m + 15b = 61 \qquad \textbf{(6)}$$

and (5) becomes

$$(1 + 2 + 3 + 4 + 5)m + 5b = 1 + 3 + 4 + 3 + 6$$

or

$$15m + 5b = 17 \qquad \textbf{(7)}$$

Solving Equation (7) for b gives

$$b = -3m + \frac{17}{5} \qquad \textbf{(8)}$$

which, upon substitution into (6), gives

$$15\left(-3m + \frac{17}{5}\right) + 55m = 61$$
$$-45m + 51 + 55m = 61$$
$$10m = 10$$
$$m = 1$$

Substituting this value of m into (8) gives

$$b = -3 + \frac{17}{5} = \frac{2}{5} = 0.4$$

Therefore, the required equation of the least-squares line is

$$y = x + 0.4$$

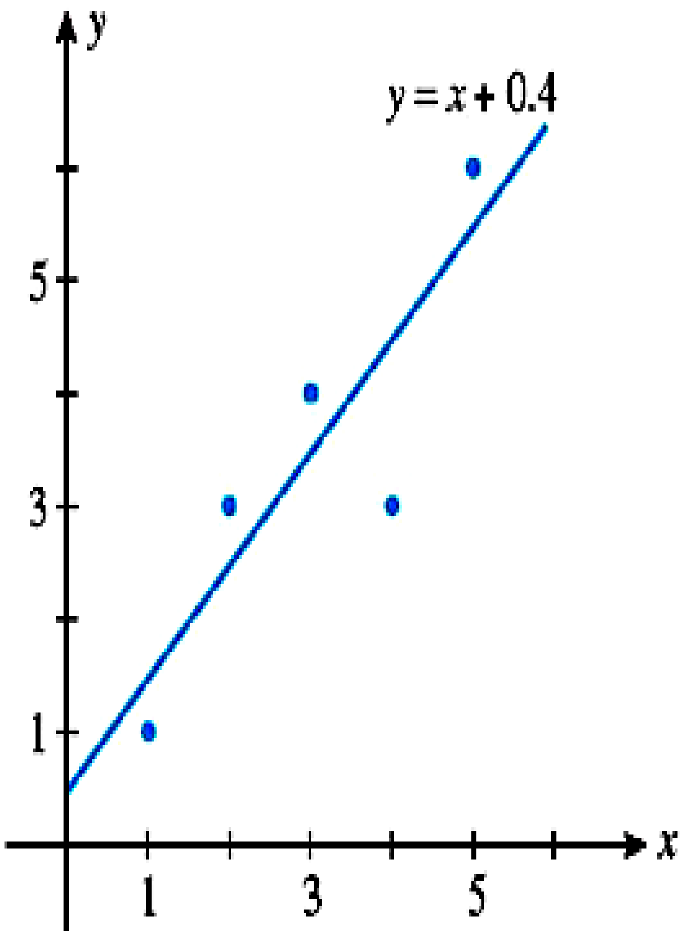


FIGURE 24
The scatter diagram and the least-squares line $y = x + 0.4$

The scatter diagram and the regression line are shown in Figure 24. ■

2012/5/27

APPLIED EXAMPLE 2 Advertising Expense and a Firm's Profit The proprietor of Leisure Travel Service compiled the following data relating the firm's annual profit to its annual advertising expenditure (both measured in thousands of dollars).

Annual Advertising Expenditure, x	12	14	17	21	26	30
Annual Profit, y	60	70	90	100	100	120

a. Determine an equation of the least-squares line for these data.
b. Draw a scatter diagram and the least-squares line for these data.
c. Use the result obtained in part (a) to predict Leisure Travel's annual profit if the annual advertising budget is $20,000.

Solution

a. The calculations required for obtaining the normal equations may be summarized as follows:

	x	y	x^2	xy
	12	60	144	720
	14	70	196	980
	17	90	289	1,530
	21	100	441	2,100
	26	100	676	2,600
	30	120	900	3,600
Sum	120	540	2,646	11,530

The normal equations are

$$6b + 120m = 540 \quad (9)$$

$$120b + 2646m = 11{,}530 \quad (10)$$

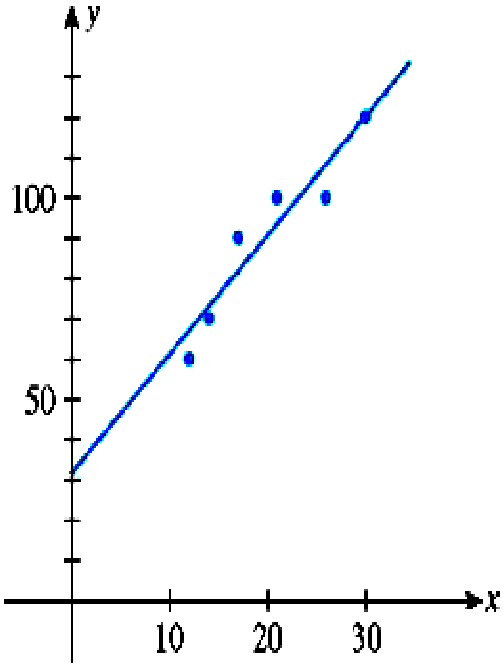


FIGURE 25
The scatter diagram and the least-squares line $y = 2.97x + 30.6$

Solving Equation (9) for b gives

$$b = -20m + 90 \qquad \textbf{(11)}$$

which, upon substitution into Equation (10), gives

$$\begin{aligned} 120(-20m + 90) + 2646m &= 11{,}530 \\ -2400m + 10{,}800 + 2646m &= 11{,}530 \\ 246m &= 730 \\ m &\approx 2.97 \end{aligned}$$

Substituting this value of m into Equation (11) gives

$$b = -20(2.97) + 90 = 30.6$$

Therefore, the required equation of the least-squares line is given by

$$y = f(x) = 2.97x + 30.6$$

b. The scatter diagram and the least-squares line are shown in Figure 25.

c. Leisure Travel's predicted annual profit corresponding to an annual budget of $20,000 is given by

$$\begin{aligned} f(20) &= 2.97(20) + 30.6 \\ &= 90 \end{aligned}$$

or $90,000. ■

APPLIED EXAMPLE 3 Maximizing Profit A market research study conducted for Century Communications provided the following data based on the projected monthly sales x (in thousands) of Century's DVD version of a box-office hit adventure movie with a proposed wholesale unit price of p dollars.

p	38	36	34.5	30	28.5
x	2.2	5.4	7.0	11.5	14.6

a. Find the demand equation if the demand curve is the least-squares line for these data.

b. The total monthly cost function associated with producing and distributing the DVD movies is given by

$$C(x) = 4x + 25$$

where x denotes the number of discs (in thousands) produced and sold and $C(x)$ is in thousands of dollars. Determine the unit wholesale price that will maximize Century's monthly profit.

Solution

a. The calculations required for obtaining the normal equations may be summarized as follows:

	x	p	x^2	xp
	2.2	38	4.84	83.6
	5.4	36	29.16	194.4
	7.0	34.5	49	241.5
	11.5	30	132.25	345
	14.6	28.5	213.16	416.1
Sum	40.7	167	428.41	1280.6

The normal equations are

$$5b + 40.7m = 167$$

$$40.7b + 428.41m = 1280.6$$

Solving this system of linear equations simultaneously, we find that

$$m \approx -0.81 \qquad \text{and} \qquad b \approx 39.99$$

Therefore, the required equation of the least-squares line is given by

$$p = f(x) = -0.81x + 39.99$$

which is the required demand equation, provided $0 \leq x \leq 49.37$.

b. The total revenue function in this case is given by

$$R(x) = xp = -0.81x^2 + 39.99x$$

and since the total cost function is

$$C(x) = 4x + 25$$

we see that the profit function is

$$\begin{aligned} P(x) &= -0.81x^2 + 39.99x - (4x + 25) \\ &= -0.81x^2 + 35.99x - 25 \end{aligned}$$

To find the absolute maximum of $P(x)$ over the closed interval [0, 49.37], we compute

$$P'(x) = -1.62x + 35.99$$

Since $P'(x) = 0$, we find $x \approx 22.22$ as the only critical point of P. Finally, from the table

x	0	22.22	49.37
$P(x)$	−25	374.78	−222.47

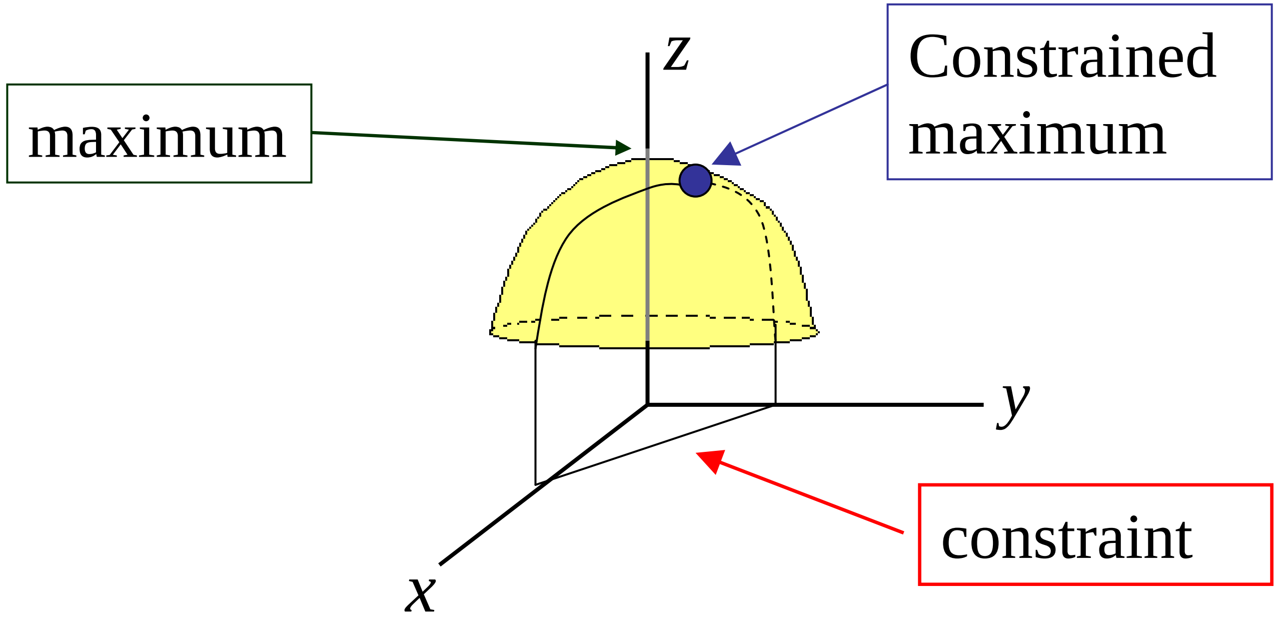
we see that the optimal wholesale price is

$$p = -0.81(22.22) + 39.99 = 21.99$$

or $21.99 per disc. ■

8.5 Constrained Maxima and Minima and Method of Lagrange Multipliers

Determining the relative extremum of a function *f* (*x*, *y*) subject to the the independent variables *x* and *y* satisfying one or more constraints, see Figure 26.



EXAMPLE 1 Find the relative minimum of the function

$$f(x, y) = 2x^2 + y^2$$

subject to the constraint $g(x, y) = x + y - 1 = 0$.

Solution Solving the constraint equation for y explicitly in terms of x, we obtain $y = -x + 1$. Substituting this value of y into the function $f(x, y) = 2x^2 + y^2$ results in a function of x,

$$h(x) = 2x^2 + (-x + 1)^2 = 3x^2 - 2x + 1$$

The function h describes the curve C lying on the graph of f on which the constrained relative minimum of f occurs. To find this point, use the technique developed in Chapter 4 to determine the relative extrema of a function of one variable:

$$h'(x) = 6x - 2 = 2(3x - 1)$$

Setting $h'(x) = 0$ gives $x = \frac{1}{3}$ as the sole critical point of the function h. Next, we find

$$h''(x) = 6$$

and, in particular,

$$h''\left(\frac{1}{3}\right) = 6 > 0$$

Therefore, by the second derivative test, the point $x = \frac{1}{3}$ gives rise to a relative minimum of h. Substitute this value of x into the constraint equation $x + y - 1 = 0$ to get $y = \frac{2}{3}$. Thus, the point $(\frac{1}{3}, \frac{2}{3})$ gives rise to the required constrained relative minimum of f. Since

$$f\left(\frac{1}{3}, \frac{2}{3}\right) = 2\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{2}{3}$$

the required constrained relative minimum value of f is $\frac{2}{3}$ at the point $(\frac{1}{3}, \frac{2}{3})$. It may be shown that $\frac{2}{3}$ is in fact a constrained absolute minimum value of f (Figure 27).

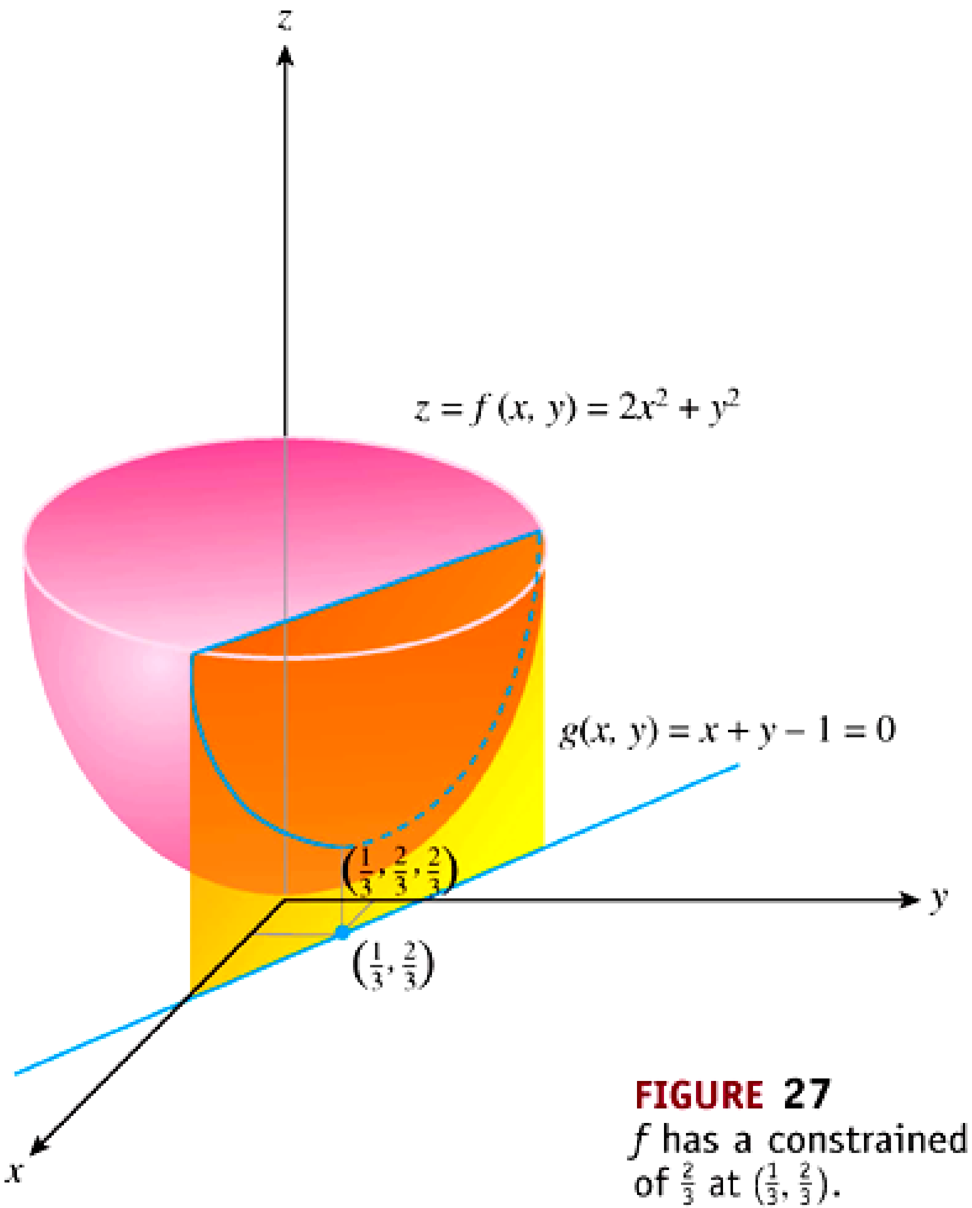


FIGURE 27
f has a constrained absolute minimum of $\frac{2}{3}$ at $\left(\frac{1}{3}, \frac{2}{3}\right)$.

The Method of Lagrange Multipliers

To find the relative extrema of the function $f(x, y)$ subject to the constraint $g(x, y) = 0$.

1. Form an auxiliary (Lagrange) function.

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

2. Solve the system:

$$F_x = 0, \quad F_y = 0, \quad F_\lambda = 0$$

3. Evaluate f at each of the points (x, y) found in step 2. The largest is the max., smallest is the min.

Ex. Use the method of Lagrange multipliers to find the constrained relative maximum of the function $f(x, y) = 1 - x^2 - y^2$ subject to $x + y = 2$.

$$F(x, y, \lambda) = 1 - x^2 - y^2 + \lambda\left(x + y - 2\right)$$

$$\left.\begin{aligned} F_x &= -2x + \lambda = 0 \\ F_y &= -2y + \lambda = 0 \\ F_\lambda &= x + y - 2 = 0 \end{aligned}\right\} \quad x = \frac{1}{2}\lambda,\ y = \frac{1}{2}\lambda \text{ and } \lambda = 2$$

F has a constrained maximum at $x = 1$, $y = 1$.

EXAMPLE 2 Using the method of Lagrange multipliers, find the relative minimum of the function

$$f(x, y) = 2x^2 + y^2$$

subject to the constraint $x + y = 1$.

Solution Write the constraint equation $x + y = 1$ in the form $g(x, y) = x + y - 1 = 0$. Then, form the Lagrangian function

$$\begin{aligned} F(x, y, \lambda) &= f(x, y) + \lambda g(x, y) \\ &= 2x^2 + y^2 + \lambda(x + y - 1) \end{aligned}$$

To find the critical point(s) of the function F, solve the system composed of the equations

$$\begin{aligned} F_x &= 4x + \lambda = 0 \\ F_y &= 2y + \lambda = 0 \\ F_\lambda &= x + y - 1 = 0 \end{aligned}$$

Solving the first and second equations in this system for x and y in terms of λ, we obtain

$$x = -\frac{1}{4}\lambda \quad \text{and} \quad y = -\frac{1}{2}\lambda$$

which, upon substitution into the third equation, yields

$$-\frac{1}{4}\lambda - \frac{1}{2}\lambda - 1 = 0 \quad \text{or} \quad \lambda = -\frac{4}{3}$$

Therefore, $x = \frac{1}{3}$ and $y = \frac{2}{3}$, and $(\frac{1}{3}, \frac{2}{3})$ affords a constrained minimum of the function f, in agreement with the result obtained earlier. ■

EXAMPLE 3 Use the method of Lagrange multipliers to find the minimum of the function

$$f(x, y, z) = 2xy + 6yz + 8xz$$

subject to the constraint

$$xyz = 12{,}000$$

(*Note:* The existence of the minimum is suggested by the geometry of the problem.)

Solution Write the constraint equation $xyz = 12{,}000$ in the form $g(x, y, z) = xyz - 12{,}000$. Then, the Lagrangian function is

$$\begin{aligned} F(x, y, z, \lambda) &= f(x, y, z) + \lambda g(x, y, z) \\ &= 2xy + 6yz + 8xz + \lambda(xyz - 12{,}000) \end{aligned}$$

To find the critical point(s) of the function F, we solve the system composed of the equations

$$\begin{aligned} F_x &= 2y + 8z + \lambda yz = 0 \\ F_y &= 2x + 6z + \lambda xz = 0 \\ F_z &= 6y + 8x + \lambda xy = 0 \\ F_\lambda &= xyz - 12{,}000 = 0 \end{aligned}$$

Solving the first three equations of the system for λ in terms of x, y, and z, we have

$$\lambda = -\frac{2y + 8z}{yz}$$
$$\lambda = -\frac{2x + 6z}{xz}$$
$$\lambda = -\frac{6y + 8x}{xy}$$

Equating the first two expressions for λ leads to

$$\frac{2y + 8z}{yz} = \frac{2x + 6z}{xz}$$
$$2xy + 8xz = 2xy + 6yz$$
$$x = \frac{3}{4}y$$

Next, equating the second and third expressions for λ in the same system yields

$$\frac{2x + 6z}{xz} = \frac{6y + 8x}{xy}$$
$$2xy + 6yz = 6yz + 8xz$$
$$z = \frac{1}{4}y$$

Finally, substituting these values of x and z into the equation $xyz - 12{,}000 = 0$, the fourth equation of the first system of equations, we have

$$\left(\frac{3}{4}y\right)(y)\left(\frac{1}{4}y\right) - 12{,}000 = 0$$

$$y^3 = \frac{(12{,}000)(4)(4)}{3} = 64{,}000$$

$$y = 40$$

The corresponding values of x and z are given by $x = \frac{3}{4}(40) = 30$ and $z = \frac{1}{4}(40) = 10$. Therefore, we see that the point (30, 40, 10) gives the constrained minimum of f. The minimum value is

$$f(30, 40, 10) = 2(30)(40) + 6(40)(10) + 8(30)(10) = 7200$$ ■

APPLIED EXAMPLE 4 Maximizing Profit Refer to Example 3, Section 8.1. The total weekly profit (in dollars) that Acrosonic realized in producing and selling its bookshelf loudspeaker systems is given by the profit function

$$P(x, y) = -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 120x + 100y - 5000$$

where x denotes the number of fully assembled units and y denotes the number of kits produced and sold per week. Acrosonic's management decides that production of these loudspeaker systems should be restricted to a total of exactly 230 units each week. Under this condition, how many fully assembled units and how many kits should be produced each week to maximize Acrosonic's weekly profit?

Solution The problem is equivalent to the problem of maximizing the function

$$P(x, y) = -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 120x + 100y - 5000$$

subject to the constraint

$$g(x, y) = x + y - 230 = 0$$

The Lagrangian function is

$$\begin{aligned} F(x, y, \lambda) &= P(x, y) + \lambda g(x, y) \\ &= -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 120x + 100y \\ &\quad - 5000 + \lambda(x + y - 230) \end{aligned}$$

To find the critical point(s) of F, solve the following system of equations:

$$F_x = -\frac{1}{2}x - \frac{1}{4}y + 120 + \lambda = 0$$
$$F_y = -\frac{3}{4}y - \frac{1}{4}x + 100 + \lambda = 0$$
$$F_\lambda = x + y - 230 = 0$$

Solving the first equation of this system for λ, we obtain

$$\lambda = \frac{1}{2}x + \frac{1}{4}y - 120$$

which, upon substitution into the second equation, yields

$$-\frac{3}{4}y - \frac{1}{4}x + 100 + \frac{1}{2}x + \frac{1}{4}y - 120 = 0$$
$$-\frac{1}{2}y + \frac{1}{4}x - 20 = 0$$

Solving the last equation for y gives

$$y = \frac{1}{2}x - 40$$

When we substitute this value of y into the third equation of the system, we have

$$x + \frac{1}{2}x - 40 - 230 = 0$$
$$x = 180$$

The corresponding value of y is $\frac{1}{2}(180) - 40$, or 50. Thus, the required constrained relative maximum of P occurs at the point (180, 50). Again, we can show that the point (180, 50) in fact yields a constrained absolute maximum for P. Thus, Acrosonic's profit is maximized by producing 180 assembled and 50 kit versions of their bookshelf loudspeaker systems. The maximum weekly profit realizable is given by

$$\begin{aligned} P(180, 50) &= -\frac{1}{4}(180)^2 - \frac{3}{8}(50)^2 - \frac{1}{4}(180)(50) \\ &\quad + 120(180) + 100(50) - 5000 \\ &= 10{,}312.5 \end{aligned}$$

or $10,312.50. ■

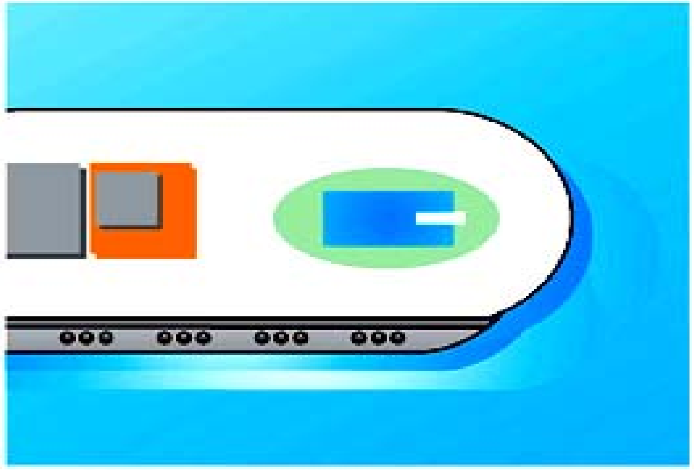
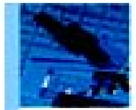


FIGURE 28
A rectangular-shaped pool will be built in the elliptical-shaped poolside area.



APPLIED EXAMPLE 5 Designing a Cruise-Ship Pool The operators of the *Viking Princess*, a luxury cruise liner, are contemplating the addition of another swimming pool to the ship. The chief engineer has suggested that an area in the form of an ellipse located in the rear of the promenade deck would be suitable for this purpose. This location would provide a poolside area with sufficient space for passenger movement and placement of deck chairs (Figure 28). It has been determined that the shape of the ellipse may be described by the equation $x^2 + 4y^2 = 3600$, where x and y are measured in feet. *Viking*'s operators would like to know the dimensions of the rectangular pool with the largest possible area that would meet these requirements.

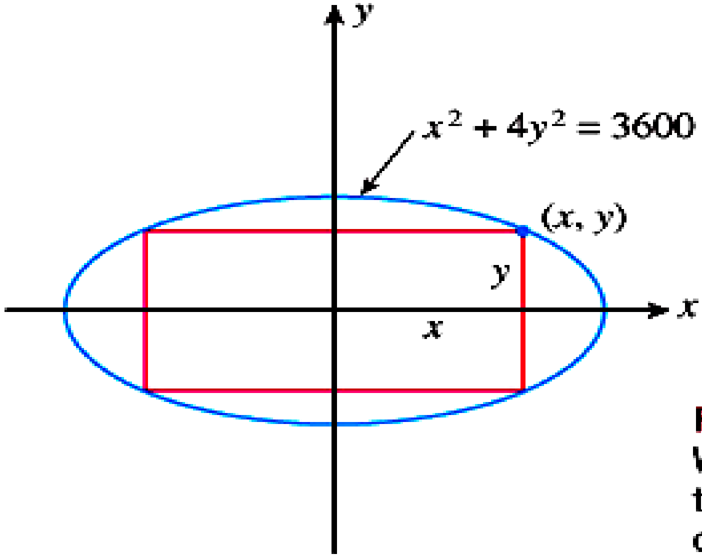


FIGURE 29
We want to find the largest rectangle that can be inscribed in the ellipse described by $x^2 + 4y^2 = 3600$.

Solution To solve this problem, we need to find the rectangle of largest area that can be inscribed in the ellipse with equation $x^2 + 4y^2 = 3600$. Letting the sides of the rectangle be $2x$ and $2y$ feet, we see that the area of the rectangle is $A = 4xy$ (Figure 29). Furthermore, the point (x, y) must be constrained to lie on the ellipse so that it satisfies the equation $x^2 + 4y^2 = 3600$. Thus, the problem is equivalent to the problem of maximizing the function

$$f(x, y) = 4xy$$

subject to the constraint $g(x, y) = x^2 + 4y^2 - 3600 = 0$. The Lagrangian function is

$$\begin{aligned} F(x, y, \lambda) &= f(x, y) + \lambda g(x, y) \\ &= 4xy + \lambda(x^2 + 4y^2 - 3600) \end{aligned}$$

To find the critical point(s) of F, we solve the following system of equations:

$$F_x = 4y + 2\lambda x = 0$$
$$F_y = 4x + 8\lambda y = 0$$
$$F_\lambda = x^2 + 4y^2 - 3600 = 0$$

Solving the first equation of this system for λ, we obtain

$$\lambda = -\frac{2y}{x}$$

which, upon substitution into the second equation, yields

$$4x + 8\left(-\frac{2y}{x}\right)y = 0 \quad \text{or} \quad x^2 - 4y^2 = 0$$

—that is, $x = \pm 2y$. Substituting these values of x into the third equation of the system, we have

$$4y^2 + 4y^2 - 3600 = 0$$

or, upon solving $y = \pm\sqrt{450} = \pm 15\sqrt{2}$. The corresponding values of x are $\pm 30\sqrt{2}$. Because both x and y must be nonnegative, we have $x = 30\sqrt{2}$ and $y = 15\sqrt{2}$. Thus, the dimensions of the pool with maximum area are $30\sqrt{2}$ feet $\times$ $60\sqrt{2}$ feet, or approximately 42 feet $\times$ 85 feet. ■

APPLIED EXAMPLE 6 Cobb–Douglas Production Function Suppose x units of labor and y units of capital are required to produce

$$f(x, y) = 100x^{3/4}y^{1/4}$$

units of a certain product (recall that this is a Cobb–Douglas production function). If each unit of labor costs \$200 and each unit of capital costs \$300 and a total of \$60,000 is available for production, determine how many units of labor and how many units of capital should be used in order to maximize production.

Solution The total cost of x units of labor at \$200 per unit and y units of capital at \$300 per unit is equal to $200x + 300y$ dollars. But \$60,000 is budgeted for production, so $200x + 300y = 60{,}000$, which we rewrite as

$$g(x, y) = 200x + 300y - 60{,}000 = 0$$

To maximize $f(x, y) = 100x^{3/4}y^{1/4}$ subject to the constraint $g(x, y) = 0$, we form the Lagrangian function

$$\begin{aligned} F(x, y, \lambda) &= f(x, y) + \lambda g(x, y) \\ &= 100x^{3/4}y^{1/4} + \lambda(200x + 300y - 60{,}000) \end{aligned}$$

To find the critical point(s) of F, we solve the following system of equations:

$$\begin{aligned} F_x &= 75x^{-1/4}y^{1/4} + 200\lambda = 0 \\ F_y &= 25x^{3/4}y^{-3/4} + 300\lambda = 0 \\ F_\lambda &= 200x + 300y - 60{,}000 = 0 \end{aligned}$$

Solving the first equation for λ, we have

$$\lambda = -\frac{75x^{-1/4}y^{1/4}}{200} = -\frac{3}{8}\left(\frac{y}{x}\right)^{1/4}$$

which, when substituted into the second equation, yields

$$25\left(\frac{x}{y}\right)^{3/4} + 300\left(-\frac{3}{8}\right)\left(\frac{y}{x}\right)^{1/4} = 0$$

Multiplying the last equation by $\left(\frac{x}{y}\right)^{1/4}$ then gives

$$25\left(\frac{x}{y}\right) - \frac{900}{8} = 0$$

$$x = \left(\frac{900}{8}\right)\left(\frac{1}{25}\right)y = \frac{9}{2}y$$

Substituting this value of x into the third equation of the first system of equations, we have

$$200\left(\frac{9}{2}y\right) + 300y - 60{,}000 = 0$$

from which we deduce that $y = 50$. Hence, $x = 225$. Thus, maximum production is achieved when 225 units of labor and 50 units of capital are used. ■

EXAMPLE 1 Let $z = f(x, y) = 2x^2 - xy$. Find Δz. Then use your result to find the change in z if (x, y) changes from (1, 1) to (0.98, 1.03).

Solution Using (12), we obtain

$$\begin{aligned}\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y)\\ &= [2(x + \Delta x)^2 - (x + \Delta x)(y + \Delta y)] - (2x^2 - xy)\\ &= 2x^2 + 4x\Delta x + 2(\Delta x)^2 - xy - x\Delta y - y\Delta x - \Delta x\Delta y - 2x^2 + xy\\ &= (4x - y)\Delta x - x\Delta y + 2(\Delta x)^2 - \Delta x\Delta y\end{aligned}$$

Next, to find the increment in z if (x, y) changes from (1, 1) to (0.98, 1.03), we note that $\Delta x = 0.98 - 1 = -0.02$ and $\Delta y = 1.03 - 1 = 0.03$. Therefore, using the result obtained earlier with $x = 1$, $y = 1$, $\Delta x = -0.02$, and $\Delta y = 0.03$, we obtain

$$\begin{aligned}\Delta z &= [4(1) - 1](-0.02) - (1)(0.03) + 2(-0.02)^2 - (-0.02)(0.03)\\ &= -0.0886\end{aligned}$$

You can verify the correctness of this result by calculating the quantity $f(0.98, 1.03) - f(1, 1)$. ■

EXAMPLE 2 Let $z = 2x^2y + y^3$.

a. Find the differential dz of z.

b. Find the approximate change in z when x changes from $x = 1$ to $x = 1.01$ and y changes from $y = 2$ to $y = 1.98$.

c. Find the actual change in z when x changes from $x = 1$ to $x = 1.01$ and y changes from $y = 2$ to $y = 1.98$. Compare the result with that obtained in part (b).

Solution

a. Let $f(x, y) = 2x^2y + y^3$. Then the required differential is

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 4xy\,dx + (2x^2 + 3y^2)\,dy$$

b. Here $x = 1$, $y = 2$, and $dx = 1.01 - 1 = 0.01$ and $dy = 1.98 - 2 = -0.02$. Therefore,

$$\Delta z \approx dz = 4(1)(2)(0.01) + [2(1) + 3(4)](-0.02) = -0.20$$

c. The actual change in z is given by

$$\begin{aligned}\Delta z &= f(1.01, 1.98) - f(1, 2)\\ &= [2(1.01)^2(1.98) + (1.98)^3] - [2(1)^2(2) + (2)^3]\\ &\approx 11.801988 - 12\\ &= -0.1980\end{aligned}$$

We see that $\Delta z \approx dz$, as expected. ■

APPLIED EXAMPLE 3 Approximating Changes in Revenue The weekly total revenue of Acrosonic Company resulting from the production and sales of x fully assembled bookshelf loudspeaker systems and y kit versions of the same loudspeaker system is

$$R(x, y) = -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 300x + 240y$$

dollars. Determine the approximate change in Acrosonic's weekly total revenue when the level of production is increased from 200 assembled units and 60 kits per week to 206 assembled units and 64 kits per week.

Solution The approximate change in the weekly total revenue is given by the total differential R at $x = 200$ and $y = 60$, $dx = 206 - 200 = 6$ and $dy = 64 - 60 = 4$; that is, by

$$\begin{aligned} dR &= \frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy \Bigg|_{\substack{x=200,\, y=60 \\ dx=6,\, dy=4}} \\ &= \left(-\frac{1}{2}x - \frac{1}{4}y + 300\right)\Bigg|_{(200,\,60)} \cdot (6) \\ &\quad + \left(-\frac{3}{4}y - \frac{1}{4}x + 240\right)\Bigg|_{(200,\,60)} \cdot (4) \\ &= (-100 - 15 + 300)6 + (-45 - 50 + 240)4 \\ &= 1690 \end{aligned}$$

or $1690.

APPLIED EXAMPLE 4 Cobb–Douglas Production Function The production for a certain country in the early years following World War II is described by the function

$$f(x, y) = 30x^{2/3}y^{1/3}$$

units, when x units of labor and y units of capital were utilized. Find the approximate change in output if the amount expended on labor had been decreased from 125 units to 123 units and the amount expended on capital had been increased from 27 to 29 units. Is your result as expected given the result of Example 4c, Section 8.2?

Solution The approximate change in output is given by the total differential of f at $x = 125$, $y = 27$, $dx = 123 - 125 = -2$, and $dy = 29 - 27 = 2$; that is, by

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \Bigg|_{\substack{x=125,\ y=27 \\ dx=-2,\ dy=2}}$$

$$= 20x^{-1/3}y^{1/3}\Big|_{(125,\ 27)} \cdot (-2) + 10x^{2/3}y^{-2/3}\Big|_{(125,\ 27)} \cdot (2)$$

$$= 20\left(\frac{27}{125}\right)^{1/3}(-2) + 10\left(\frac{125}{27}\right)^{2/3}(2)$$

$$= -20\left(\frac{3}{5}\right)(2) + 10\left(\frac{25}{9}\right)(2) = \frac{284}{9}$$

or $31\frac{5}{9}$ units. This result is fully compatible with the result of Example 4, where the recommendation was to encourage increased spending on capital rather than on labor.

APPLIED EXAMPLE 5 **Error Analysis** Find the maximum percentage error in calculating the volume of a rectangular box if an error of at most 1% is made in measuring the length, width, and height of the box.

Solution Let x, y, and z denote the length, width, and height, respectively, of the rectangular box. Then the volume of the box is given by $V = f(x, y, z) = xyz$ cubic units. Now suppose the true dimensions of the rectangular box are a, b, and c units, respectively. Since the error committed in measuring the length, width, and height of the box is at most 1%, we have

$$|\Delta x| = |x - a| \leq 0.01a$$
$$|\Delta y| = |y - b| \leq 0.01b$$
$$|\Delta z| = |z - c| \leq 0.01c$$

Therefore, the maximum error in calculating the volume of the box is

$$\begin{aligned}
|\Delta V| \approx |dV| &= \left|\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right|\Bigg|_{x=a,\, y=b,\, z=c} \\
&= |yz\,dx + xz\,dy + xy\,dz|\Big|_{x=a,\, y=b,\, z=c} \\
&= |bc\,dx + ac\,dy + ab\,dz| \\
&\leq bc|dx| + ac|dy| + ab|dz| \\
&\leq bc(0.01a) + ac(0.01b) + ab(0.01c) \\
&= (0.03)abc
\end{aligned}$$

Since the actual volume of the box is abc cubic units, we see that the maximum percentage error in calculating its volume is

$$\frac{|\Delta V|}{V|_{(a,b,c)}} \approx \frac{(0.03)abc}{abc} = 0.03$$

—that is, approximately 3%. ■

8.7 Double Integral

The **double integral of** $f(x, y)$ **over the region** R is denoted

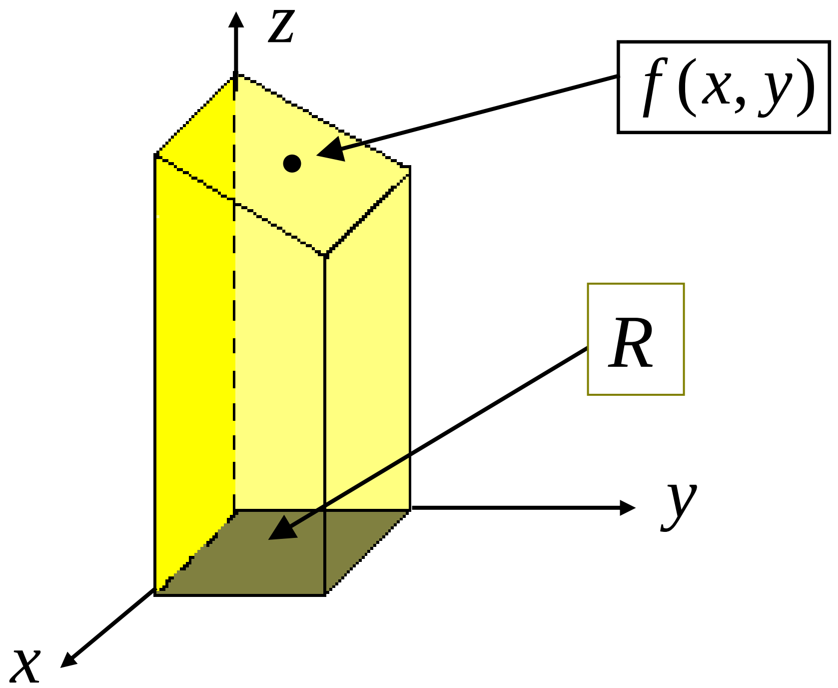
$$\iint_R f(x, y)\,dA$$

If $f(x, y)$ is nonnegative, then the integral gives the volume of the solid bounded above by $z = f(x, y)$ and below by the plane region R.

Evaluating a Double Integral Over a Rectangular Region (x and y are not related)

Let R be defined by $a \le x \le b$ and $c \le y \le d$

Then $$\iint_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$



Ex. Evaluate $\iint_R f(x,y)\,dA$, where $f(x,y) = 3x^2 + 4y$

and R is the rectangle defined by

$1 \le x \le 2$ and $2 \le y \le 4$

$$\iint_R f(x,y)\,dA = \int_2^4 \left[\int_1^2 \left(3x^2 + 4y\right) dx \right] dy$$

$$= \int_2^4 \left[\left(x^3 + 4xy\right)\Big|_1^2 \right] dy = \int_2^4 (7 + 4y)\, dy$$

$$= \left(7y + 2y^2\right)\Big|_2^4 = 60 - 22 = 38$$

EXAMPLE 1 Evaluate $\iint_R f(x, y)\, dA$, where $f(x, y) = x + 2y$ and R is the rectangle defined by $1 \le x \le 4$ and $1 \le y \le 2$.

Solution Using Equation (15), we find

$$\iint_R f(x, y)\, dA = \int_1^2 \left[\int_1^4 (x + 2y)\, dx \right] dy$$

To compute

$$\int_1^4 (x + 2y)\, dx$$

we treat y as if it were a constant (remember that dx reminds us that we are integrating with respect to x). We obtain

$$\begin{aligned} \int_1^4 (x + 2y)\, dx &= \frac{1}{2}x^2 + 2xy \bigg|_{x=1}^{x=4} \\ &= \left[\frac{1}{2}(16) + 2(4)y\right] - \left[\frac{1}{2}(1) + 2(1)y\right] \\ &= \frac{15}{2} + 6y \end{aligned}$$

Thus,

$$\begin{aligned} \iint_R f(x, y)\, dA &= \int_1^2 \left(\frac{15}{2} + 6y\right) dy = \left(\frac{15}{2}y + 3y^2\right)\bigg|_1^2 \\ &= (15 + 12) - \left(\frac{15}{2} + 3\right) = 16\tfrac{1}{2} \end{aligned}$$

■

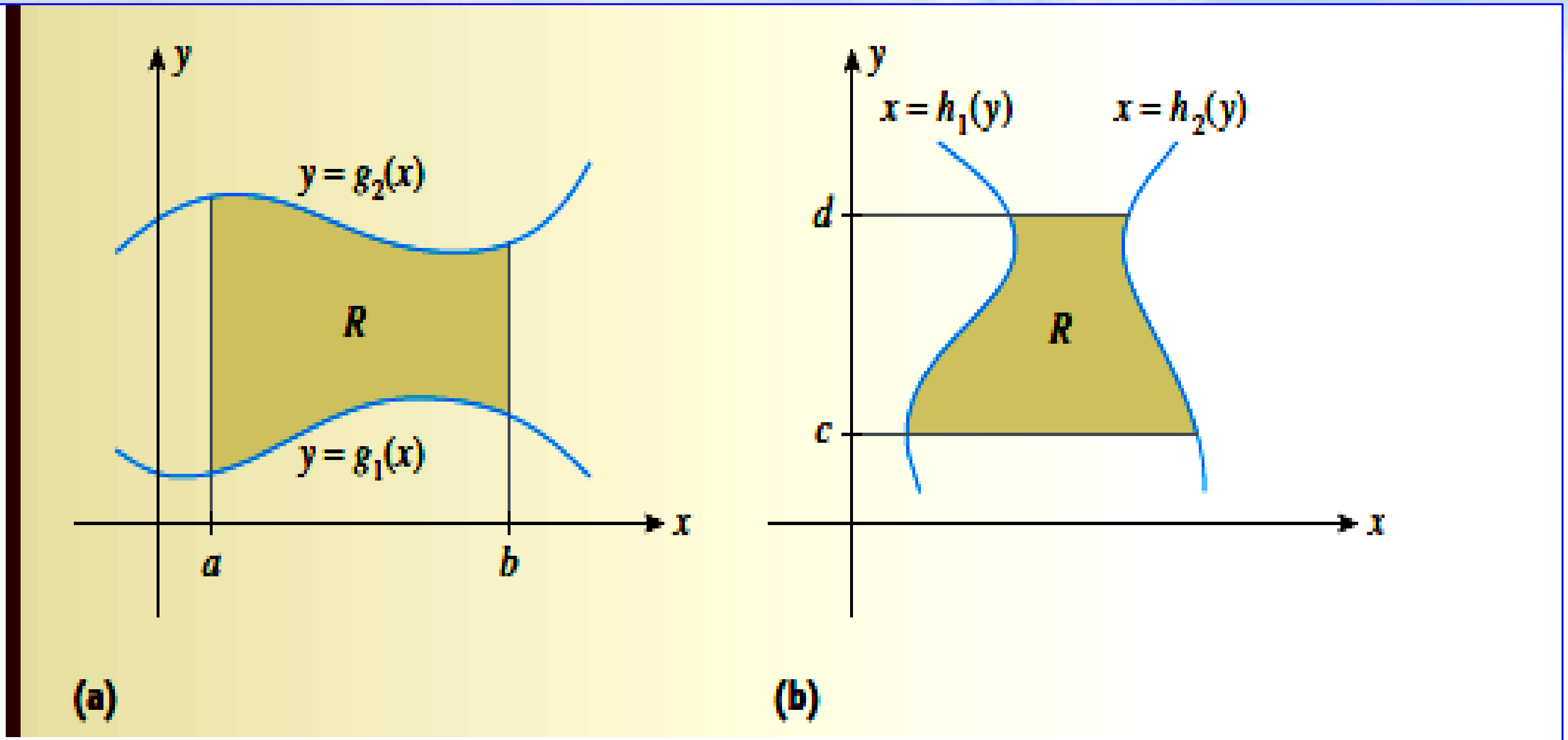
Evaluating a Double Integral Over a Plane Region (x and y are related)

Suppose $g_1(x)$ and $g_2(x)$ are continuous on $[a, b]$ and $R = \{(x, y) \mid g_1(x) \le y \le g_2(x); a \le x \le b\}$. Then

$$\iint_R f(x, y)\,dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\,dy \right] dx$$

Suppose $h_1(y)$ and $h_2(y)$ are continuous on $[c, d]$ and $R = \{(x, y) \mid h_1(y) \le x \le h_2(y); c \le y \le d\}$. Then

$$\iint_R f(x, y)\,dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y)\,dx \right] dy$$


y
y = g2(x)
R
y = g1(x)
x
a
b
(a)
y
x = h1(y)
x = h2(y)
d
R
c
x
(b)

Ex. Evaluate $\iint_R (x+2y)\,dA$ where R is the region bounded by $y=x^2$ and $y=2-x^2$.

$$\iint_R (x+2y)\,dA = \int_{-1}^{1}\left[\int_{x^2}^{2-x^2} (x+2y)\,dy\right]dx$$

$$= \int_{-1}^{1}\left[\left(xy+y^2\right)\Big|_{x^2}^{2-x^2}\right]dx$$

$$= \int_{-1}^{1}\left(-2x^3-4x^2+2x+4\right)dx$$

$$= \left(-\frac{1}{2}x^4-\frac{4}{3}x^3+x^2+4x\right)\Big|_{-1}^{1}$$

$$= \frac{19}{6}-\left(\frac{-13}{6}\right)=\frac{16}{3}$$

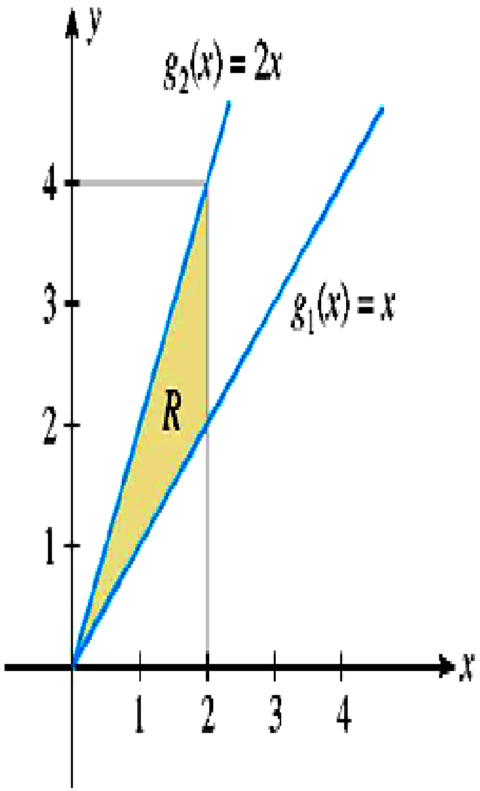


FIGURE 37
R is the region bounded by $g_1(x) = x$ and $g_2(x) = 2x$ for $0 \leq x \leq 2$.

EXAMPLE 2 Evaluate $\iint_R f(x, y)\, dA$ given that $f(x, y) = x^2 + y^2$ and R is the region bounded by the graphs of $g_1(x) = x$ and $g_2(x) = 2x$ for $0 \leq x \leq 2$.

Solution The region under consideration is shown in Figure 37. Using Equation (16), we find

$$\iint_R f(x, y)\, dA = \int_0^2 \left[\int_x^{2x} (x^2 + y^2)\, dy \right] dx$$

$$= \int_0^2 \left[\left(x^2 y + \frac{1}{3} y^3 \right) \Bigg|_x^{2x} \right] dx$$

$$= \int_0^2 \left[\left(2x^3 + \frac{8}{3} x^3 \right) - \left(x^3 + \frac{1}{3} x^3 \right) \right] dx$$

$$= \int_0^2 \frac{10}{3} x^3\, dx = \frac{5}{6} x^4 \Bigg|_0^2 = 13\tfrac{1}{3}$$

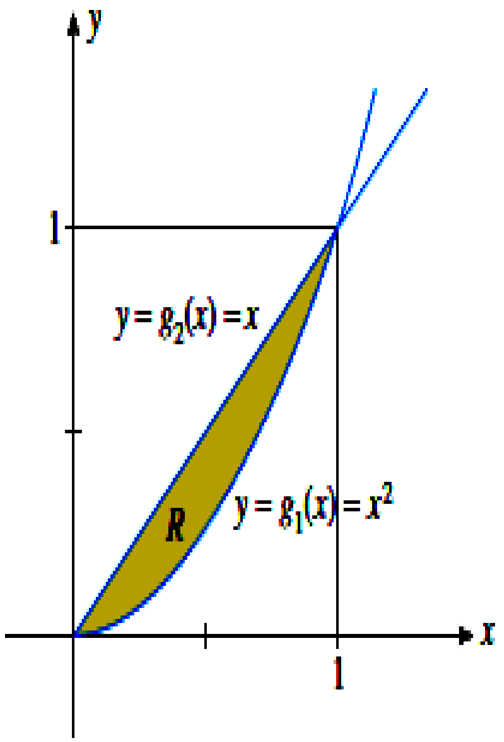


FIGURE 38
R is the region bounded by $y = x^2$ and $y = x$.

EXAMPLE 3 Evaluate $\iint_R f(x, y)\, dA$, where $f(x, y) = xe^y$ and R is the plane region bounded by the graphs of $y = x^2$ and $y = x$.

Solution The region in question is shown in Figure 38. The point of intersection of the two curves is found by solving the equation $x^2 = x$, giving $x = 0$ and $x = 1$. Using Equation (16), we find

$$\iint_R f(x, y)\, dA = \int_0^1 \left[\int_{x^2}^{x} xe^y\, dy \right] dx = \int_0^1 \left[xe^y \Big|_{x^2}^{x} \right] dx$$

$$= \int_0^1 (xe^x - xe^{x^2})\, dx = \int_0^1 xe^x\, dx - \int_0^1 xe^{x^2}\, dx$$

and integrating the first integral on the right-hand side by parts,

$$= \left[(x - 1)e^x - \frac{1}{2} e^{x^2} \right] \Bigg|_0^1$$

$$= -\frac{1}{2} e - \left(-1 - \frac{1}{2} \right) = \frac{1}{2}(3 - e)$$

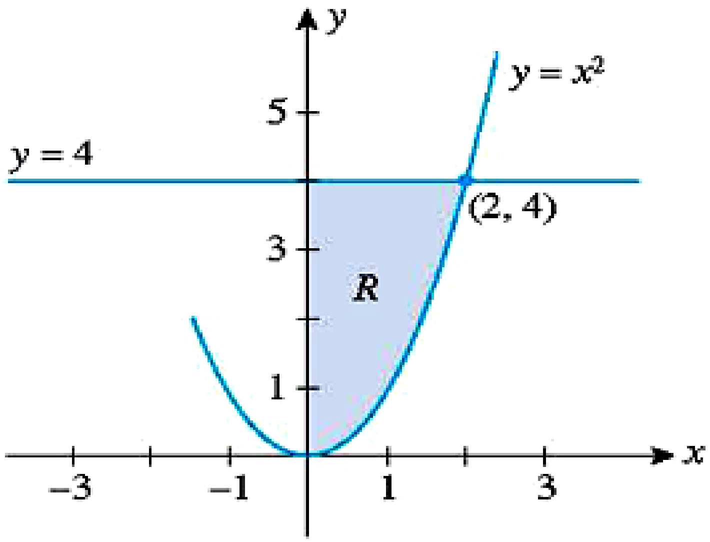


FIGURE 39
R is the region bounded by the y-axis, $x = 0$, $y = 4$, and $y = x^2$.

EXAMPLE 4 Evaluate

$$\iint_R xe^{y^2}\,dA$$

where R is the plane region bounded by the y-axis, $x = 0$, the horizontal line $y = 4$, and the graph of $y = x^2$.

Solution The region R is shown in Figure 39. The point of intersection of the line $y = 4$ and the graph of $y = x^2$ is found by solving the equation $x^2 = 4$, giving $x = 2$ and the required point (2, 4). Using Equation (16) with $y = g_1(x) = x^2$ and $y = g_2(x) = 4$ leads to

$$\iint_R xe^{y^2}\,dA = \int_0^2 \left[\int_{x^2}^4 xe^{y^2}\,dy \right] dx$$

Now evaluation of the integral

$$\int_{x^2}^{4} xe^{y^2}\,dy = x\int_{x^2}^{4} e^{y^2}\,dy$$

calls for finding the antiderivative of the integrand e^{y^2} in terms of elementary functions, a task that, as was pointed out in Section 7.3, cannot be done. Let's begin afresh and attempt to make use of Equation (17).

Since the equation $y = x^2$ is equivalent to the equation $x = \sqrt{y}$, which clearly expresses x as a function of y, we may write, with $x = h_1(y) = 0$ and $h_2(y) = \sqrt{y}$,

$$\iint_R xe^{y^2}\,dA = \int_0^4\left[\int_0^{\sqrt{y}} xe^{y^2}\,dx\right]dy = \int_0^4\left[\frac{1}{2}x^2e^{y^2}\Big|_0^{\sqrt{y}}\right]dy$$

$$= \int_0^4 \frac{1}{2}ye^{y^2}\,dy = \frac{1}{4}e^{y^2}\Big|_0^4 = \frac{1}{4}(e^{16}-1) \quad \blacksquare$$

8.8 The Volume of a Solid under a Surface

Let R be the region in the xy-plane and let f be continuous and nonnegative on R. Then, the volume of the solid bounded above by the surface $z = f(x, y)$ and below by R is given by

$$V = \iint_R f(x, y)\, dA$$

Example

Find the volume of the solid bounded above by the plane $z = f(x, y) = y$ and below by the plane region R defined by $y = \sqrt{1 - x^2} \quad (0 \le x \le 1)$.

$$V = \iint_R f(x, y)\, dA = \iint_R y \, dA$$

$$= \int_0^1 \left[\int_0^{\sqrt{1-x^2}} y \, dy \right] dx = \int_0^1 \left[\frac{1}{2} y^2 \Big|_0^{\sqrt{1-x^2}} \right] dx$$

$$= \int_0^1 \left[\frac{1}{2}\left(1 - x^2\right) \right] dx = \frac{1}{2}\left(x - \frac{1}{3} x^3 \right)\Big|_0^1 = \frac{1}{3}$$

EXAMPLE 1 Find the volume of the solid bounded above by the plane $z = f(x, y) = y$ and below by the plane region R defined by $y = \sqrt{1 - x^2}$ $(0 \le x \le 1)$.

Solution The region R is sketched in Figure 40. Observe that $f(x, y) = y \ge 0$ for $(x, y) \in R$. Therefore, the required volume is given by

$$\iint_R y\, dA = \int_0^1 \left[\int_0^{\sqrt{1-x^2}} y\, dy \right] dx = \int_0^1 \left[\frac{1}{2} y^2 \Big|_0^{\sqrt{1-x^2}} \right] dx$$

$$= \int_0^1 \frac{1}{2}(1 - x^2)\, dx = \frac{1}{2}\left(x - \frac{1}{3}x^3 \right)\Big|_0^1 = \frac{1}{3}$$

or $\frac{1}{3}$ cubic unit. The solid is shown in Figure 41. Note that it is not necessary to make a sketch of the solid in order to compute its volume.

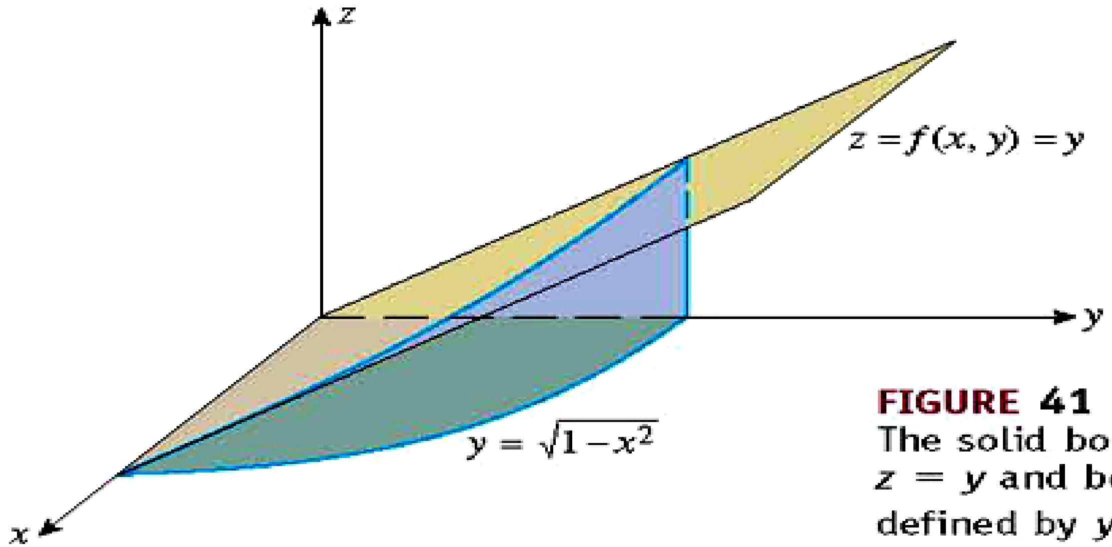


FIGURE 41
The solid bounded above by the plane $z = y$ and below by the plane region defined by $y = \sqrt{1 - x^2}$ $(0 \le x \le 1)$

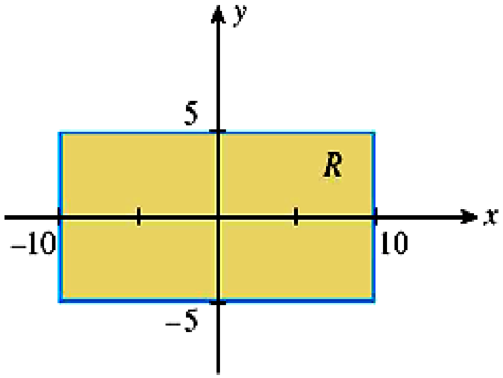


FIGURE 43
The rectangular region R represents a certain district of a city.

APPLIED EXAMPLE 2 Population Density The population density of a certain city is described by the function

$$f(x, y) = 10{,}000e^{-0.2|x|-0.1|y|}$$

where the origin (0, 0) gives the location of the city hall. What is the population inside the rectangular area described by

$$R = \{(x, y) | -10 \le x \le 10;\ -5 \le y \le 5\}$$

if x and y are in miles? (See Figure 43.)

Solution By symmetry, it suffices to compute the population in the first quadrant. (Why?) Then, upon observing that in this quadrant

$$f(x, y) = 10{,}000e^{-0.2x-0.1y} = 10{,}000e^{-0.2x}e^{-0.1y}$$

we see that the population in R is given by

$$\begin{aligned}\iint_R f(x, y)\, dA &= 4\int_0^{10}\left[\int_0^5 10{,}000e^{-0.2x}e^{-0.1y}\, dy\right] dx \\ &= 4\int_0^{10}\left[-100{,}000e^{-0.2x}e^{-0.1y}\Big|_0^5\right] dx \\ &= 400{,}000(1 - e^{-0.5})\int_0^{10} e^{-0.2x}\, dx \\ &= 2{,}000{,}000(1 - e^{-0.5})(1 - e^{-2})\end{aligned}$$

or approximately 680,438 people.

Average Value of $f(x, y)$ Over the Region R

If f is integrable over the plane region R, then its *average value* over R is given by

$$\frac{\iint\limits_R f(x, y)\,dA}{\text{Area of } R} = \frac{\iint\limits_R f(x, y)\,dA}{\iint\limits_R dA}$$

Ex. Find the average value of $f(x,y)=2x+y^2$ over the region bounded by $y=x^2$, the x-axis, $x=0$ and $x=1$.

$$\int_0^1\left[\int_0^{x^2}\left(2x+y^2\right)dy\right]dx=\int_0^1\left[\left(2xy+\frac{y^3}{3}\right)_0^{x^2}\right]dx$$

$$=\int_0^1\left(2x^3+\frac{x^6}{3}\right)dx=\left(\frac{1}{2}x^4+\frac{1}{21}x^7\right)\Big|_0^1=\frac{23}{42}$$

Since $\underbrace{\int_0^1 x^2dx=\frac{1}{3}}_{\text{Area of region}}$, the average value of f is

$$\frac{23/42}{1/3}=\frac{23}{14}$$

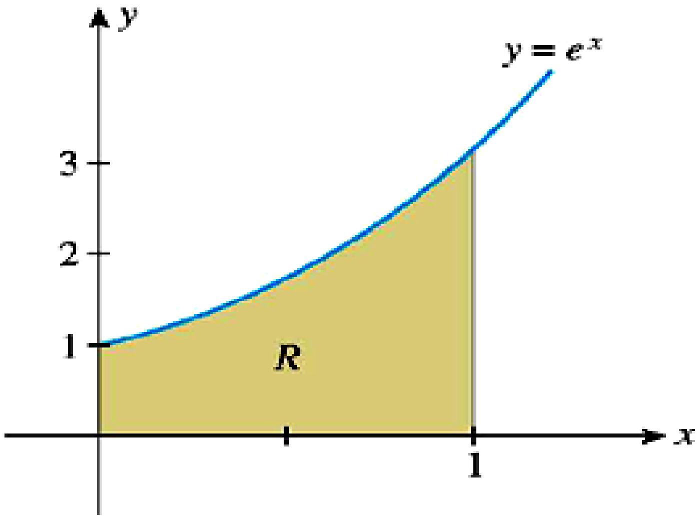


FIGURE 44
The plane region R defined by $y = e^x$ $(0 \le x \le 1)$

EXAMPLE 3 Find the average value of the function $f(x, y) = xy$ over the plane region defined by $y = e^x$ $(0 \le x \le 1)$.

Solution The region R is shown in Figure 44. The area of the region R is given by

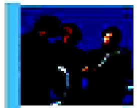
$$\begin{aligned}\int_0^1 \left[\int_0^{e^x} dy \right] dx &= \int_0^1 \left[y \Big|_0^{e^x} \right] dx \\ &= \int_0^1 e^x \, dx \\ &= e^x \Big|_0^1 \\ &= e - 1\end{aligned}$$

square units. We would obtain the same result had we viewed the area of this region as the area of the region under the curve $y = e^x$ from $x = 0$ to $x = 1$. Next, we compute

$$\begin{aligned}
\iint_R f(x, y)\, dA &= \int_0^1 \left[\int_0^{e^x} xy\, dy \right] dx \\
&= \int_0^1 \left[\frac{1}{2} xy^2 \Big|_0^{e^x} \right] dx \\
&= \int_0^1 \frac{1}{2} xe^{2x}\, dx \\
&= \frac{1}{4} xe^{2x} - \frac{1}{8} e^{2x} \Big|_0^1 \qquad \text{Integrate by parts.} \\
&= \left(\frac{1}{4} e^2 - \frac{1}{8} e^2 \right) + \frac{1}{8} \\
&= \frac{1}{8}(e^2 + 1)
\end{aligned}$$

square units. Therefore, the required average value is given by

$$\frac{\displaystyle\iint_R f(x, y)\, dA}{\displaystyle\iint_R dA} = \frac{\frac{1}{8}(e^2 + 1)}{e - 1} = \frac{e^2 + 1}{8(e - 1)}$$



APPLIED EXAMPLE 4 Population Density (Refer to Example 2.) The population density of a certain city (number of people per square mile) is described by the function

$$f(x, y) = 10{,}000e^{-0.2|x| - 0.1|y|}$$

where the origin gives the location of the city hall. What is the average population density inside the rectangular area described by

$$R = \{(x, y) | -10 \le x \le 10; -5 \le y \le 5\}$$

where x and y are measured in miles?

Solution From the results of Example 2, we know that

$$\iint_R f(x, y)\, dA \approx 680{,}438$$

From Figure 40, we see that the area of the plane rectangular region R is (20)(10), or 200, square miles. Therefore, the average population inside R is

$$\frac{\displaystyle\iint_R f(x, y)\, dA}{\displaystyle\iint_R dA} = \frac{680{,}438}{200} = 3402.19$$

or approximately 3402 people per square mile. ■